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“Indefinite Integral”

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Indefinite Integral

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This study guide aims to develop as rapidly as possible the student's ability to understand and to use the constitutive part of integral calculus "Indefinite Integral". The comprehensive analysis of methods of integration is given. The detailed solutions of a typical tasks are shown. More than 20 variants of individual tasks on different topics of the course is represented. This study guide is focused on the organization of system training and self-training.

Integral calculus is an essential tool for engineers which students must use from the very beginning of their studies. That is why this textbook is designed for the first- and second-years students of all specialties of technical higher education institutions that learn mathematics in English).

Навчальний посібник містить основні теоретичні положення стосовно розділу "Невизначений інтеграл" курсу вищої математики. Дано вичерпний аналіз основним способам знаходження невизначеного інтегралу. Наведено вичерпні розв'язки достатньої кількості типових задач. Містить понад 20 варіантів індивідуальних завдань з різних тем курсу «невизначений інтеграл». Орієнтований на організацію системної підготовки та самопідготовки.

Розраховано на студентів перших та других курсів всіх спеціальностей технічних вищих навчальних закладів денної, вечірньої, заочної та дистанційної форм навчання, іноземних студентів, та студентів спеціальностей з викладанням математики англійською мовою.

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Indefinite Integral

Tutorial

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CONTENT

Introduction

Highly-qualified specialist training for various fields of science and manufacturing is the most important goal of the modern technical research university. Good command of foreign languages is one of the main factors of successful scientific and practical activity of a modern specialist. Studying English is one of the main ways to integrate students, as future specialists, into the international system of production, technology and economics. In this regard, it is necessary to enhance the system of students' teaching in foreign languages, English in particular, and thus, integrate them into the international educational environment.

Therefore, teaching English requires developing some methodological recommendations. This is especially important in the process of learning mathematics because studying any of the technical disciplines is impossible without mathematics background. The purpose of this textbook is to improve the quality and forecast study results of foreign students as well as the students delivered the course of Higher Mathematics in English. Also, this textbook is designed to increase the effectiveness of learners'self-study.

The content of the textbook "An Uncertain Integral" covers the general course of Higher Mathematics for technical specialties. A fairly complete and concise presentation of the theoretical information is introduced at the beginning of each section. Then, detailed solutions of typical examples arranged in the order of increased complexity are given and proved. A number of typical tasks are thourougly analyzed in this textbook empowering students' opportunity to solve independently individual tasks presented at the end of each section.

Studying the material of this textbook allows a student to learn how to work independently and obtain sufficient practical skills required not only for studying other sections of Higher Mathematics but other special disciplines as well.

The textbook is intended for students of the first and second courses of all specialties of technical higher education institutions of full-time, correspondent and distant forms of study.

§1. Primitive Function and Indefinite Integral

1.1. ***The Primitive Function.*** The function $F(x)$ is a primitive function of $f(x)$ in some interval X if the following holds true:

$$F'(x) = f(x), \quad \forall x \in X. \tag{1}$$

Obviously, the definition (1) makes sense only when the function $F(x)$ is differentiable in the interval X.

Example 1.

a) Let $f(x) = 2x$. The primitive function is $F(x) = x^2$ in $X = (-\infty, \infty)$. We can easily verify the result by differentiating: $F'(x) = \left(x^2\right)' = 2x = f(x) \quad \forall x \in X$;

b) Let $f(x) = \sin 3x$. It is easy to find a function $F(x)$, satisfying equality (1):

$$F(x) = -\frac{1}{3}\cos 3x,\ x \in X = (-\infty, \infty).$$

Indeed, $F'(x) = \left(-\frac{1}{3}\cos 3x\right)' = -\frac{1}{3}(-3\sin 3x) = \sin 3x = f(x), \quad \forall x \in X.$

Remember that the derivative of a constant term is zero ($C' = 0$). Hence, if a constant term is added to the function $F(x)$, the function obtained will also have the same derivative. Therefore, we can put: $[F(x) + C]' = F'(x) = f(x)$.

Let $f(x) = 2x$. Then the primitive functions will be $F(x) = x^2$ and $F_1(x) = x^2 + 1$ and $F_2(x) = x^2 - 3$, etc. It obviously follows that there is not just one solution but many others which differ only by constants: $F(x) + C$.

The primitive of a function $f(x)$ without any additional conditions is always uncertain. If the function $f(x)$ in the interval X has the primitive function $F(x)$ then it also has an infinite set of primitive functions $F(x)+C$, where C = const.

1.2. ***Indefinite Integral.*** The set of all primitive functions of a function $f(x)$ in the interval X is called the indefinite integral of the function $f(x)$.

This is written as:

$$\int f(x)\,dx = F(x) + C. \tag{2}$$

Here $f(x)$ – integrand; $f(x)dx$ – an element of integration; x – variable of integration; C – arbitrary constant; $\int$ – the integral sign.

For example, $\int 2x\,dx = x^2 + C$.

Equality y = $F(x) + C$ determines some curve with a fixed C.

When C is changing, the integral curve shifts along the Oy axis. Thus, all primitive functions differ from each other by a constant which can be determined from specified boundary conditions (fig. 1).

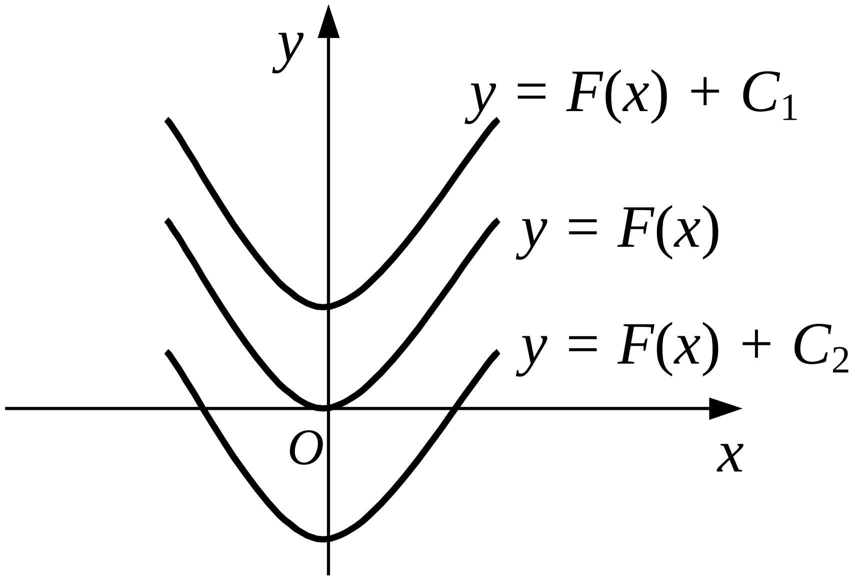

Fig. 1

1.3. ***Indefinite integral properties.***

1. Derivative of indefinite integral is the integrand, and the differential of indefinite integral is the element of integration:

$$\left(\int f(x)dx\right)' = f(x), \qquad d\int f(x)dx = f(x)dx. \tag{3}$$

2. Indefinite integral of the differential of a function is the sum of this function and an arbitrary constant:

$$\int dF(x) = F(x) + C. \tag{4}$$

3. If A is a **constant** and $A \neq 0$, then

$$\int A f(x)dx = A\int f(x)dx. \tag{5}$$

4. The integral of the sum of two or more functions is equal to the sum of the integrals of the individual functions:

$$\int\left[f(x) \pm g(x)\right]dx = \int f(x)dx \pm \int g(x)dx. \tag{6}$$

1.4. ***Table of Fundamental Standard Integrals.*** The Table of **standard integrals** contains some elementary functions and their integrals. It is easy to verify the results by differentiation according to (3).

Table1. Fundamental Standard Integrals

1. $\int du = u + C$	**8**. $\int ctgu\, du = \ln\lvert \sin u\rvert + C$
2. $\int u^{\alpha} du = \dfrac{u^{\alpha+1}}{\alpha+1} + C,\ \alpha \neq -1$	**9**. $\int \dfrac{du}{\cos^2 u} = \int \sec^2 u\, du = tgu + C$
2*a*. $\int \dfrac{du}{\sqrt{u}} = 2\sqrt{u} + C$	**10**. $\int \dfrac{du}{\sin^2 u} = \int \operatorname{cosec}^2 u\, du = -\operatorname{ctg} u + C$
2*б*. $\int \dfrac{du}{u^2} = -\dfrac{1}{u} + C$	**11**. $\int \dfrac{du}{\sin u} = \ln\left\lvert tg\dfrac{u}{2}\right\rvert + C$
3. $\int \dfrac{du}{u} = \ln\lvert u\rvert + C$	**12**. $\int \dfrac{du}{\cos u} = \ln\left\lvert \operatorname{tg}\left(\dfrac{u}{2} + \dfrac{\pi}{4}\right)\right\rvert + C$
4. $\int a^u\, du = \dfrac{a^u}{\ln a} + C$	**13**. $\int \dfrac{du}{u^2 + a^2} = \dfrac{1}{a}\operatorname{arctg}\dfrac{u}{a} + C$
4*a*. $\int e^u\, du = e^u + C$	**14**. $\int \dfrac{du}{u^2 - a^2} = \dfrac{1}{2a}\ln\left\lvert\dfrac{u-a}{u+a}\right\rvert + C$
5. $\int \sin u\, du = -\cos u + C$	**15**. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin\dfrac{u}{a} + C$
6. $\int \cos u\, du = \sin u + C$	**16**. $\int \dfrac{du}{\sqrt{u^2 \pm a^2}} = \ln\left\lvert u + \sqrt{u^2 \pm a^2}\right\rvert + C$
7. $\int \operatorname{tg} u\, du = -\ln\lvert\cos u\rvert + C$	**17**. $\int u\, dv = uv - \int v\, du$

§ 2. Techniques of Integration

2.1. ***Direct and substitution integration.*** Some integrands after algebraic or trigonometric transformations could be integrated just using the table of standard integrals.

Example 1.

$$\int (2x^5 + \frac{3}{x^4} - 5\sqrt[4]{x} - \frac{6}{\sqrt[7]{x}})dx.$$

According to the properties (4) and (3) this integral can be represented in such a way:

$$2\int x^5 dx + 3\int \frac{dx}{x^4} - 5\int \sqrt[4]{x} dx - 6\int \frac{dx}{\sqrt[7]{x}} = 2\int x^5 dx + 3\int x^{-4} dx \text{ - } 5\int x^{\frac{1}{4}} dx - 6\int x^{-\frac{1}{7}} dx.$$

Then using formula (2) one can obtain:

$$2\cdot\frac{x^6}{6}+3\cdot\frac{x^{-3}}{-3}-5\cdot\frac{x^{\frac{5}{4}}}{\frac{5}{4}}-6\cdot\frac{x^{\frac{6}{7}}}{\frac{6}{7}}+C=\frac{x^6}{3}-\frac{1}{x^3}-4\cdot x^{\frac{5}{4}}-7\cdot x^{\frac{6}{7}}+C$$

Example 2.

$\int\left(7x^9-3\cos x+5e^x\right)dx.$

According to the properties (4) and (3) this integral can be represented in such a way:

$7\int x^9dx\ -3\int\cos xdx\ +\ 5\int e^xdx.$

Then using formulas (2), (6) and (4a) obtain:

$7\cdot\frac{x^{10}}{10}\ -3\sin x\ +\ 5e^x+C.$

Example 3.

$\int\frac{dx}{\sin^2 x\cos^2 x}\ .$

We can transform the integrand using fundamental trigonometric formulas:

$$\frac{1}{\sin^2 x\cos^2 x}=\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}=\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x},$$

therefore, $I=\int\frac{dx}{\cos^2 x}+\int\frac{dx}{\sin^2 x}=I_1+I_2$. Integrals I_1 and I_2 should be found by formulas 9 and 10 of standard integrals table. So, $I = tgx - ctgx + C$.

In general, the integral $I(x)=\int f(x)dx$ may not belong to the tabular type, but it can be transformed using substitution: $f(x)dx=f\left[x(u)\right]x'(u)du=\phi(u)du$, and then it could be integrated by the table: $I(x)=\int\phi(u)du=I_1(u)$.

In particular, if $u = ax + b$, where a and b are constants, the following holds are true:

$$\int f(ax+b)d(ax+b)=F(ax+b)+C \tag{7}$$

or

$$a\int f(ax+b)dx=F(ax+b)+C.$$

Therefore

$$\int f(ax+b)dx=\frac{1}{a}F(ax+b)+C. \tag{8}$$

Following example demonstrates the formula (8) usage:

а) $\int \sin x\,dx = -\cos x + C \quad \Rightarrow \quad \int \sin(3x+1)\,dx = -\frac{1}{3}\cos(3x+1) + C$;

б) $\int e^x dx = e^x + C \quad \Rightarrow \quad \int e^{-2x} dx = -\frac{1}{2}e^{-2x} + C$.

Try following examples as well.

Example 4.

$$\int \left(\frac{7}{25+4x^2} \right) dx.$$

Let us use the formula (8) and formula (13) from Table 1:

$$\int \left(\frac{7}{5^2+(2x)^2} \right) dx = \frac{1}{10} arctg\,\frac{2x}{5} + C.$$

Example 5.

$$\int \left(\frac{7}{\sqrt{5-3x}} + 2\sqrt{\frac{x}{5}+4} - \frac{1}{\sin^2(3x-1)} \right) dx.$$

According to the properties (4) and (3) this integral can be represented in such a way:

$$\int \left(\frac{7}{\sqrt{5-3x}} + 2\sqrt{\frac{x}{5}+4} - \frac{1}{\sin^2(3x-1)} \right) dx = 7\int \frac{dx}{\sqrt{5-3x}} + 2\int \sqrt{\frac{x}{5}+4}\,dx - \int \frac{dx}{\sin^2(3x-1)}.$$

Now let us use the formula (8) and formulas (2) and (10) from Table 1:

$$\frac{7}{-3}\cdot 2 \cdot \sqrt{5-3x} + 2\cdot 5 \cdot \frac{2}{3} \cdot \sqrt{\left(\frac{x}{5}+4\right)^3} + \frac{1}{3} ctg(3x-1) + C =$$

$$= -\frac{14}{3}\cdot \sqrt{5-3x} + \frac{20}{3} \cdot \sqrt{\left(\frac{x}{5}+4\right)^3} + \frac{1}{3} ctg(3x-1) + C.$$

Example 6.

Let us return to integral $\int \frac{dx}{\sin^2 x \cos^2 x}$ (example 3).

We can apply another method for this example. As you know, $\sin 2x = 2\sin x \cos x$ then $\frac{1}{\sin^2 x \cos^2 x} = \frac{4}{\sin^2 2x}$, so,

$$I = 4\int \frac{dx}{\sin^2 2x} = \text{(using formula (8))} = -\frac{4}{2} ctg\,2x + C = -2ctg\,2x + C.$$

To verify the results, we should make some transformation

$$-2ctg\,2x = -2\frac{\cos 2x}{\sin 2x} = -2\frac{\cos^2 x - \sin^2 x}{2\sin x \cos x} = -\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = tgx - ctgx.$$

Hence, the results of examples (6) and (3) coincide.

Individual Task 1

Variant 1

1. $\int (x+1)^2 dx$	2. $\int \left(x^2+2\cos x+e^x\right)dx$	3. $\int \frac{(\sqrt{x}+1)^2}{3x}dx$
4. $\int \left(\frac{2}{\sqrt{16-x^2}}+\frac{5}{\cos^2 x}\right)dx$	5. $\int \left(\frac{7}{\sqrt{x}}+2\sqrt{x}-\frac{1}{\sin^2 x}\right)dx$	6. $\int \left((\sqrt[5]{x}+1)^2-\frac{5^x}{3}\right)dx$
7. $\int \left(5\sqrt[3]{x^2}+2ctgx+\frac{1}{3x}\right)dx$	8. $\int \left(\frac{5}{x^3}+\frac{7}{5+x^2}\right)dx$	9. $\int \left(5\sin x-\frac{3}{x}\right)dx$
10. $\int \cos(5-2x)dx$	11. $\int \frac{1}{3x-7}dx$	12. $\int \sin(1-\frac{x}{8})dx$

Variant 2

1. $\int (2x-1)^2 dx$	2. $\int \left(x^3+3\cos x+2^x\right)dx$	3. $\int \frac{(\sqrt{x}-3)^2}{2x^2}dx$
4. $\int \left(\frac{5}{x^2}-\frac{3}{\sqrt{5-x^2}}\right)dx$	5. $\int \left(\frac{3}{4+x^2}-\frac{e^x}{2}\right)dx$	6. $\int \left(4\cos x+\frac{2}{x}\right)dx$
7. $\int \left(\frac{3}{\sqrt{9-x^2}}-\frac{1}{\cos^2 x}\right)dx$	8. $\int \left(\frac{2}{\sqrt[4]{x}}-\sqrt{x}+\frac{3}{\cos^2 x}\right)dx$	9. $\int \left(\frac{3}{\sqrt{4+x^2}}-\frac{tgx}{2}\right)dx$
10. $\int e^{(3+2x)}dx$	11. $\int \frac{1}{\sqrt{5x-2}}dx$	12. $\int \cos(4-\frac{x}{7})dx$

Variant 3

1. $\int (2-3x)^2 dx$	2. $\int \left(3x-2\cos x+4^x\right)dx$	3. $\int \frac{(\sqrt[3]{x}+5)^2}{2x}dx$
4. $\int \left(\frac{4}{x}-\frac{3}{\sqrt{16-x^2}}\right)dx$	5. $\int \left(\frac{1}{3+x^2}-\frac{e^x}{2}\right)dx$	6. $\int \left(2tgx-\frac{7}{\sqrt{x}}\right)dx$
7. $\int \left(\frac{2}{\sqrt{4+x^2}}+\frac{4}{\sin^2 x}\right)dx$	8. $\int \left(\frac{2}{\sqrt[4]{x}}-4x-\frac{1}{2\cos^2 x}\right)dx$	9. $\int \left(\frac{2}{\sqrt{x}}-\frac{3}{x}-\frac{1}{2\cos x}\right)dx$
10. $\int 6^{3x+2}dx$	11. $\int \frac{1}{\cos^2(4x+6)}dx$	12. $\int \sin(\frac{x}{6}+1)dx$

Variant 4

1. $\int (1+3x)^2 dx$	2. $\int \left(3+x^3-2ctgx+3^x\right)dx$	3. $\int \frac{(\sqrt{x}+1)^2}{5x^3}dx$
4. $\int \left(\frac{1}{x}-\frac{3}{6-x^2}\right)dx$	5. $\int \left(\frac{7}{7+x^2}+\frac{5e^x}{3}\right)dx$	6. $\int \left(\frac{2}{5}\cos x-\frac{7}{x^2}\right)dx$
7. $\int \left(\frac{2}{\sqrt[4]{x}}-4x-\frac{1}{2\cos^2 x}\right)dx$	8. $\int \left(\frac{2}{x}+\frac{3}{\cos^2 x}\right)dx$	9. $\int \left(\frac{2}{\sqrt{9-x^2}}+\frac{3}{\sin x}\right)dx$
10. $\int (6x+11)^5 dx$	11. $\int \frac{1}{\sin^2(4-3x)}dx$	12. $\int e^{\frac{x}{6}+1}dx$

Variant 5

1. $\int (2+x)^3 dx$	2. $\int \left(3x+\frac{4}{x^3}-tgx+3e^x\right)dx$	3. $\int \frac{(\sqrt[4]{x}-2)^2}{2x^2}dx$
4. $\int \left(1-\frac{3}{6+x^2}\right)dx$	5. $\int \left(\frac{1}{\sqrt{7+x^2}}-\frac{e^x}{3}\right)dx$	6. $\int \left(\frac{1}{5}\sin x-\frac{3}{\sqrt{x}}\right)dx$
7. $\int \left(\frac{2}{\sqrt[3]{x}}-3+\frac{1}{5\sin^2 x}\right)dx$	8. $\int \left(\frac{3}{x\sqrt{x}}-\frac{2}{\cos^2 x}\right)dx$	9. $\int \left(\frac{3}{\sqrt{1-x^2}}-\frac{2}{\cos x}\right)dx$
10. $\int \frac{1}{(3x-2)^5}dx$	11. $\int \cos(5-\frac{2x}{7})dx$	12. $\int 7^{\frac{x}{5}+8}dx$

Variant 6

1. $\int (1-x)^3 dx$	2. $\int \frac{(\sqrt[3]{x}+1)^2}{4\sqrt{x}}dx$	3. $\int \left(5-\frac{1}{2}\sin x-\frac{3}{x}\right)dx$
4. $\int \left(\frac{3}{x^2\sqrt{x}}-4tgx\right)dx$	5. $\int \left(3x^2-\frac{4}{\sqrt{x^3}}+ctgx+3^x\right)dx$	6. $\int \left(\frac{1}{2\sqrt{x}}-1+\frac{4}{\cos^2 x}\right)dx$
7. $\int \left(4x^3-\frac{5}{25+x^2}\right)dx$	8. $\int \left(\frac{1}{\sqrt{49+x^2}}+\frac{e^x}{7}\right)dx$	9. $\int \left(\frac{3}{\sqrt{9-x^2}}+\frac{\cos x}{7}\right)dx$
10. $\int \frac{1}{\sqrt[3]{3x-2}}dx$	11. $\int tg(\frac{3x}{2}-1)dx$	12. $\int \frac{1}{\cos^2(3-x)}dx$

Variant 7

1. $\int (x+1)^3 dx$	2. $\int \left(3x^2-2\cos x+\frac{8}{x}\right)dx$	3. $\int \frac{(\sqrt{x}+1)^3}{3x}dx$
4. $\int \left(\frac{5}{x^6}+\frac{5}{25+x^2}\right)dx$	5. $\int \left((\sqrt[3]{x}+1)^2-\frac{7^x}{3}\right)dx$	6. $\int \left(5ctgx-\frac{3}{x^4}\right)dx$

7. $\int\left(\frac{7}{x\sqrt{x}}-2\sqrt{x}-\frac{6}{\sin^2 x}\right)dx$	8. $\int\left(\frac{4}{\sqrt{4-x^2}}-\frac{3}{\cos^2 x}\right)dx$	9. $\int\left(\frac{2}{\sqrt{9+x^2}}-\frac{5}{\cos x}\right)dx$
10. $\int\frac{1}{3x+5}dx$	11. $\int\cos(\frac{8x}{5}-2)dx$	12. $\int ctg(\frac{x}{3}+9)dx$

Variant 8

1. $\int(5-2x)^2dx$	2. $\int\left(2-x^3+\frac{3}{x}+2^x\right)dx$	3. $\int\frac{(2\sqrt{x}-1)^2}{3x^2}dx$
4. $\int\left(\frac{1}{5x^2}+\frac{3}{\sqrt{25-x^2}}\right)dx$	5. $\int\left(\frac{4}{16+x^2}-4e^x\right)dx$	6. $\int\left(2\sin x-\frac{1}{2\sqrt{x}}\right)dx$
7. $\int\left(\frac{4}{3-x^2}+\frac{1}{2\cos^2 x}\right)dx$	8. $\int\left(\frac{2}{\sqrt[6]{x}}-5\sqrt{x}-\frac{3}{\sin^2 x}\right)dx$	9. $\int\left(\frac{4}{\sqrt{9-x^2}}+\frac{1}{2\cos x}\right)dx$
10. $\int\frac{1}{\cos^2(3+5x)}dx$	11. $\int\sin(\frac{3x}{8}-7)dx$	12. $\int\frac{1}{3-5x}dx$

Variant 9

1. $\int(3-x)^2dx$	2. $\int\left(3x-2\cos x+\frac{5}{x}\right)dx$	3. $\int\frac{(\sqrt[3]{x}-1)^2}{5x}dx$
4. $\int\left(1+\frac{1}{4x}-\frac{3}{\sqrt{4-x^2}}\right)dx$	5. $\int\left(\frac{5}{5+x^2}+\frac{e^x}{5}\right)dx$	6. $\int\left(2-tgx+\frac{2}{\sqrt{x}}\right)dx$
7. $\int\left(\frac{2}{\sqrt[7]{x}}-7x-\frac{1}{7\cos^2 x}\right)dx$	8. $\int\left(\frac{5}{\sqrt{1+x^2}}+\frac{7}{\sin^2 x}\right)dx$	9. $\int\left(\frac{5}{\sqrt{16+x^2}}+\frac{7}{\sin x}\right)dx$
10. $\int\frac{1}{\cos^2(5-2x)}dx$	11. $\int\frac{1}{7x-3}dx$	12. $\int\sin(5-\frac{x}{9})dx$

Variant 10

1. $\int(x+2)^3dx$	2. $\int\left(x^2+1-\frac{2}{5}tgx+5^x\right)dx$	3. $\int\frac{(x\sqrt{x}+1)^2}{4x^2}dx$
4. $\int\left(\frac{1}{\sqrt{x}}-\frac{3}{\sqrt{6-x^2}}\right)dx$	5. $\int\left(\frac{7}{\sqrt{7+x^2}}+\frac{5e^x}{3}\right)dx$	6. $\int\left(x-2\cos x-\frac{1}{2x^2}\right)dx$
7. $\int\left(\frac{1}{3x}-\frac{7}{\cos^2 x}\right)dx$	8. $\int\left(\frac{2}{\sqrt[4]{x}}-4x-\frac{1}{2\sin^2 x}\right)dx$	9. $\int\left(\frac{2}{x\sqrt{x}}-4-\frac{1}{2\sin x}\right)dx$
10. $\int e^{3x+2}dx$	11. $\int\frac{1}{\sqrt{5-2x}}dx$	12. $\int\cos(7-\frac{x}{4})dx$

Variant 11

1. $\int (2-3x)^2 dx$	2. $\int \left(3x-7x^3-2\sin x+e^x\right)dx$	3. $\int \frac{(\sqrt[3]{x}-3)^2}{2\sqrt{x}}dx$
4. $\int \left(\frac{1}{x\sqrt{x}}-\frac{3}{\sqrt{9+x^2}}\right)dx$	5. $\int \left(\frac{2}{49+x^2}+\frac{e^x}{3}\right)dx$	6. $\int \left(\frac{2}{5}-\cos x-\frac{1}{5\sqrt{x}}\right)dx$
7. $\int \left(2+\frac{1}{2x}+\frac{4}{\cos^2 x}\right)dx$	8. $\int \left(\frac{2}{\sqrt[3]{x}}-4x-ctgx\right)dx$	9. $\int \left(2x+\frac{1}{2\sqrt{x}}+\frac{4}{\cos x}\right)dx$
10. $\int 3^{4x+7}dx$	11. $\int \frac{1}{\cos^2(4-6x)}dx$	12. $\int \sin(\frac{2x}{5}+4)dx$

Variant 12

1. $\int (1-2x)^3 dx$	2. $\int \left(5x+\frac{6}{x^2}-2ctgx+3^x\right)dx$	3. $\int \frac{(3\sqrt{x}-1)^2}{2x}dx$
4. $\int \left(6-\frac{3}{36+x^2}\right)dx$	5. $\int \left(\frac{1}{\sqrt{9+x^2}}-\frac{2e^x}{3}\right)dx$	6. $\int \left(\frac{1}{2}\sin x-\frac{3}{\sqrt[3]{x}}\right)dx$
7. $\int \left(\frac{2}{x^2\sqrt{x}}+\frac{1}{4\cos^2 x}\right)dx$	8. $\int \left(\frac{1}{4\sqrt[4]{x}}-3+\frac{1}{2\sin^2 x}\right)dx$	9. $\int \left(\frac{1}{4x}+\frac{1}{2\sin x}\right)dx$
10. $\int (4x+1)^6 dx$	11. $\int \frac{1}{\sin^2(3-7x)}dx$	12. $\int e^{\frac{x}{2}+3}dx$

Variant 13

1. $\int (2+x)^3 dx$	2. $\int \left(2x^3-\frac{1}{4\sqrt{x^3}}+8ctgx+4^x\right)dx$	3. $\int \frac{(\sqrt[3]{x}+1)^3}{4\sqrt{x}}dx$
4. $\int \left(4x^3-\frac{5}{\sqrt{25+x^2}}\right)dx$	5. $\int \left(\frac{1}{4+x^2}+\frac{e^x}{2}\right)dx$	6. $\int \left(7-\frac{1}{3}\sin x+\frac{3}{x}\right)dx$
7. $\int \left(\frac{5}{x\sqrt{x}}-\frac{1}{2}tgx\right)dx$	8. $\int \left(\frac{x}{2\sqrt{x}}-x+\frac{4}{5\cos^2 x}\right)dx$	9. $\int \left(9-x+\frac{4}{5\cos x}\right)dx$
10. $\int \frac{1}{(2x-1)^3}dx$	11. $\int \cos(2-\frac{3x}{4})dx$	12. $\int 5^{\frac{2x}{7}+8}dx$

Variant 14

1. $\int (x-1)^3 dx$	2. $\int \left(4x^3-\cos x+\frac{1}{2x}\right)dx$	3. $\int \frac{(\sqrt{x}+3)^3}{2x}dx$

4. $\int\left(\frac{1}{2x\sqrt{x}}+2\sqrt{x}-\frac{5}{\sin^2 x}\right)dx$	5. $\int\left((2\sqrt[3]{x}-1)^2-\frac{4^x}{3}\right)dx$	6. $\int\left(2-3ctgx-\frac{2}{x^3}\right)dx$
7. $\int\left(\frac{3}{\sqrt{9-x^2}}-\frac{3}{\cos^2 x}\right)dx$	8. $\int\left(\frac{5}{x^6}-\frac{5}{25+x^2}\right)dx$	9. $\int\left(\frac{3}{\sqrt{9+x^2}}-\frac{3}{\cos x}\right)dx$
10. $\int\frac{1}{\sqrt[4]{4x-5}}dx$	11. $\int tg(\frac{2x}{5}+1)dx$	12. $\int\frac{1}{\cos^2(3x-7)}dx$

Variant 15

1. $\int\frac{(2x-3)^3}{x}dx$	2. $\int\left(2-3x^3-\frac{2}{3}\cos x+\frac{8}{x}\right)dx$	3. $\int\frac{(\sqrt{x}+x)^3}{3x^2}dx$
4. $\int\left(\frac{2}{x^5}-x+\frac{6}{\cos^2 x}\right)dx$	5. $\int\left((\sqrt{x}+1)^2+6e^x\right)dx$	6. $\int\left(x+5tgx-\frac{3}{x^5}\right)dx$
7. $\int\left(\frac{5}{9-x^2}-\frac{1}{2\cos^2 x}\right)dx$	8. $\int\left(\frac{1}{5x^2}-\frac{15}{3+x^2}\right)dx$	9. $\int\left(\frac{5}{\sqrt{9-x^2}}-\frac{1}{2\cos x}\right)dx$
10. $\int\frac{1}{4x+5}dx$	11. $\int\cos(\frac{2x}{5}+2)dx$	12. $\int ctg(\frac{3x}{5}+9)dx$

Variant 16

1. $\int x(2-3x)^2dx$	2. $\int\left(3x^3-2+\frac{4}{x}+2^x\right)dx$	3. $\int\frac{(3-2\sqrt{x})^2}{2x}dx$
4. $\int\left(\frac{4}{4+x^2}-4e^x\right)dx$	5. $\int\left(\frac{8}{x^2}+\frac{9}{\sqrt{4-x^2}}\right)dx$	6. $\int\left(x-2\sin x+\frac{1}{2\sqrt{x}}\right)dx$
7. $\int\left(\frac{3}{4-x^2}+\frac{1}{3\cos^2 x}\right)dx$	8. $\int\left(\frac{2}{\sqrt[3]{x}}-7\sqrt{x}+\frac{4}{\sin^2 x}\right)dx$	9. $\int\left(2-7x\sqrt{x}+\frac{4}{\sin\ x}\right)dx$
10. $\int\frac{1}{\sqrt{3x+5}}dx$	11. $\int\sin(\frac{4x}{7}-3)dx$	12. $\int tg(\frac{7x}{3}+2)dx$

Variant 17

1. $\int\frac{(1-x)^2}{2x}dx$	2. $\int\left(3+x-\cos x+\frac{7}{x}\right)dx$	3. $\int\frac{(\sqrt{x}-1)^2}{2x}dx$
4. $\int\left(\frac{2}{3\sqrt{x}}-x-\frac{9}{\cos^2 x}\right)dx$	5. $\int\left(\frac{1}{4x}-\frac{3}{\sqrt{9-x^2}}\right)dx$	6. $\int\left(1-2tgx+\frac{3}{2\sqrt{x}}\right)dx$
7. $\int\left(\frac{4}{\sqrt{2+x^2}}+\frac{3}{2\sin^2 x}\right)dx$	8. $\int\left(\frac{2}{4+x^2}+\frac{e^x}{2}\right)dx$	9. $\int\left(\frac{4}{\sqrt{2-x^2}}+\frac{3}{2\sin x}\right)dx$

10. $\int \frac{1}{\cos^2(3-7x)}dx$	11. $\int \sin(\frac{8x}{3}-4)dx$	12. $\int \frac{1}{1-3x}dx$

Variant 18

1. $\int \frac{(x-2)^3}{5x}dx$	2. $\int \left(\frac{3}{2}-4x^2-\frac{2}{5}tgx+5^x\right)dx$	3. $\int \frac{(x^2\sqrt{x}-1)^2}{4x^2}dx$
4. $\int \left(\frac{8}{\sqrt{x}}-\frac{3}{\sqrt{6+x^2}}\right)dx$	5. $\int \left(\frac{7}{\sqrt{7-x^2}}+\frac{e^x}{3}\right)dx$	6. $\int \left(\frac{x}{9}-2\cos x-\frac{8}{x^3}\right)dx$
7. $\int \left(\frac{2}{\sqrt[3]{x}}-4x^3-\frac{2}{\sin^2 x}\right)dx$	8. $\int \left(\frac{7}{3x}-\frac{9}{\cos^2 x}\right)dx$	9. $\int \left(7-x-\frac{9}{\cos x}\right)dx$
10. $\int \frac{1}{\cos^2(2-5x)}dx$	11. $\int \frac{1}{\sqrt[3]{7x-3}}dx$	12. $\int \sin(4-\frac{2x}{5})dx$

Variant 19

1. $\int (3-x)^2 dx$	2. $\int \left(x+4x^3-\frac{1}{2}\sin x+e^x\right)dx$	3. $\int \frac{(\sqrt{x}-2)^3}{3x}dx$
4. $\int \left(\frac{5}{x^6}-\frac{5}{25+x^2}\right)dx$	5. $\int \left((2\sqrt[4]{x}-1)^2-\frac{4^x}{3}\right)dx$	6. $\int \left(\frac{2}{5}-\cos x-\frac{1}{5\sqrt{x}}\right)dx$
7. $\int \left(\frac{3}{\sqrt{9-x^2}}-\frac{3}{\cos^2 x}\right)dx$	8. $\int \left(\frac{1}{2x\sqrt{x}}+2\sqrt{x}-4tgx\right)dx$	9. $\int \left(\frac{3}{\sqrt{9+x^2}}-\frac{3}{\cos x}\right)dx$
10. $\int \frac{1}{\sin^2(7-3x)}dx$	11. $\int \frac{1}{(7x-3)^5}dx$	12. $\int \sin(4-\frac{7x}{12})dx$

Variant 20

1. $\int (1-2x)^3 dx1$	2. $\int \left(4x^3-\frac{1}{4\sqrt{x^2}}+7ctgx+3^x\right)dx$	3. $\int \frac{(\sqrt[3]{x}-1)^3}{3\sqrt{x}}dx$
4. $\int \left(\frac{2}{2+x^2}+\frac{e^x}{2}\right)dx$	5. $\int \left(4x^3-\frac{7}{\sqrt{49+x^2}}\right)dx$	6. $\int \left(8-\frac{1}{5}\sin x+\frac{4}{x}\right)dx$
7. $\int \left(\frac{3}{x\sqrt{x}}-\frac{1}{2}ctgx\right)dx$	8. $\int \left(\frac{x^3}{2\sqrt{x}}-4x+\frac{1}{5\cos^2 x}\right)dx$	9. $\int \left(\frac{1}{2\sqrt{x^2+1}}-\frac{1}{5\cos x}\right)dx$
10. $\int \frac{1}{\cos^2(5x+2)}dx$	11. $\int \frac{1}{(7x-3)^2}dx$	12. $\int \sin(1-\frac{2x}{3})dx$

Let us return to the mentioned above transformation (8) of an integrand using new variable $u = u(x)$:

$$f(x)dx = f\left[x(u)\right]x'(u)du = \phi(u)du$$

that leads to the tabular integral type:

$$I(x) = \int \phi(u)du = I_1(u).$$

In general case new variable $u = u(x)$ should be chosen to simplify the expression by substitution of this variable and its differential in the integral. So, if the integrand contains factor $u'(x)$ then $u = u(x)$ will be a good substitution. A right selection of new variable should transform the integral to the tabular form.

If substitution is successful, the integral I_1 could be solved easily. Then we need to express the results in terms of original variable x.

Example 7.

$$\int \frac{\sqrt{1+\ln x}}{x}dx = \left\{ \begin{matrix} u = 1+\ln x \\ du = \frac{1}{x}dx \end{matrix} \right\} = \int \sqrt{u}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}\sqrt{(1+\ln x)^3} + C.$$

Example 8.

$$\int \frac{x\,dx}{\sqrt{x^2+1}} = \frac{1}{2}\int \frac{2x\,dx}{\sqrt{x^2+1}} = \frac{1}{2}\int \frac{d\left(x^2+1\right)}{\sqrt{x^2+1}} = \left\{u = x^2+1\right\} = \frac{1}{2}\int \frac{du}{\sqrt{u}} =$$

$$= \frac{1}{2}\cdot 2\sqrt{u} + C = \sqrt{x^2+1} + C.$$

Example 9.

$$\int \frac{\sin 2x\,dx}{1+\cos 2x} = \left\{ \begin{matrix} u = 1+\cos 2x \\ du = -2\sin 2x\,dx \end{matrix} \right\} = -\frac{1}{2}\int \frac{-2\sin 2x\,dx}{1+\cos 2x} =$$

$$= -\frac{1}{2}\int \frac{d(1+\cos 2x)}{1+\cos 2x} = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln(1+\cos 2x) + C.$$

Example 10.

$$\int \frac{x\,dx}{x^4+3} = \int \frac{x\,dx}{\left(x^2\right)^2+3} = \left\{ \begin{matrix} u = x^2 \\ du = 2x\,dx \end{matrix} \right\} = \frac{1}{2}\int \frac{2x\,dx}{\left(x^2\right)^2+3} =$$

$$= \frac{1}{2}\int \frac{du}{u^2+3} = \frac{1}{2\sqrt{3}}arctg\frac{u}{\sqrt{3}} + C = \frac{1}{2\sqrt{3}}arctg\frac{x^2}{\sqrt{3}} + C.$$

Integral $I(x) = \int f(x)dx$ can be transformed to the tabular type using such substitution as well:

$$I(x) = \int f(x)dx = \left\{ \begin{matrix} x = \varphi(t) \\ dx = \varphi'(t) \end{matrix} \right\} = \int f(\varphi(t))\varphi'(t)dt = \int \Phi(t)dt = I_1(t). \quad (9)$$

Here $t = \psi(x)$ is a function inverse to $x = \phi(t)$ and $I(x) = I_1[\psi(x)]$.

We can verify the result by differentiation ($\phi'(t) \neq 0$):

$$\left(\int f(x)dx\right)'_x = f(x) \text{ and } \left(\int f[\phi(t)]\phi'(t)dt\right)'_x = \left(\int f[\phi(t)]\phi'(t)dt\right)'_t \cdot \frac{dt}{dx} =$$

$$= f(\phi(t))\phi(t) \cdot \frac{1}{dx/dt} = f(\phi(t))\phi(t) \cdot \frac{1}{\phi'(t)} = f(\phi(t)) = f(x).$$

Example 11.

$$\int \frac{x\,dx}{\sqrt{x+1}} = \left\{\begin{matrix} x = t^2 - 1, & t = \sqrt{x+1} \\ dx = 2t\,dt & \end{matrix}\right\} = \int \frac{\left(t^2-1\right)\cdot 2t\,dt}{t} =$$

$$= 2\int\left(t^2-1\right)dt = 2\left(\int t^2 dt - \int dt\right) = 2\left(\frac{t^3}{3} - t\right) + C = \frac{2}{3}\sqrt{(x+1)^3} - 2\sqrt{x+1} + C.$$

Example 12.

$$\int \frac{dx}{1+\sqrt[3]{x^2}} = \left\{\begin{matrix} x = t^3, \; t = \sqrt[3]{x} \\ dx = 3t^2\,dt \end{matrix}\right\} = \int \frac{3t^2\,dt}{1+t^2} = 3\int \frac{\left(t^2+1\right)-1}{t^2+1}dt =$$

$$= 3\int\left(1 - \frac{1}{t^2+1}\right)dt = 3\left(\int dt - \int \frac{dt}{t^2+1}\right) = 3t - 3arctgt + C = 3\sqrt[3]{x} - 3arctg\sqrt[3]{x} + C$$

Individual Task 2

Variant 1

1. $\int \frac{1}{x(\ln x + 2)}dx$	2. $\int \frac{\sin x}{(1+\cos x)}dx$	3. $\int \frac{\sin 2x}{1-\cos 2x}dx$
4. $\int \frac{x}{\sqrt{5-x^2}}dx$	5. $\int \frac{arctg\,2x}{1+4x^2}dx$	6. $\int \frac{1}{x(2\ln x + 1)}dx$
7. $\int x \sin x^2 dx$	8. $\int \frac{e^{x/5}}{1-2e^{x/5}}dx$	9. $\int x\left(\sqrt{3-x^2}\right)^3 dx$

Variant 2

1. $\int \frac{1}{x\sqrt{\ln x + 5}}dx$	2. $\int \frac{\cos x}{(7+\sin x)}dx$	3. $\int \frac{\sqrt[3]{tgx}}{\cos^2 x}dx$
4. $\int x\sqrt[5]{x^2+4}dx$	5. $\int \frac{arctg^3 x}{1+x^2}dx$	6. $\int \frac{1}{x(\ln^2 x + 4)}dx$
7. $\int xtgx^2 dx$	8. $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$	9. $\int \frac{\arcsin x}{\sqrt{1-x^2}}dx$

Variant 3

1. $\int \frac{\sqrt{\ln x+3}}{x}dx$	2. $\int \frac{\cos x}{\sqrt{2-\sin x}}dx$	3. $\int \frac{\sqrt[5]{ctgx}}{\sin^2 x}dx$
4. $\int \frac{x}{\sqrt[3]{x^2+2}}dx$	5. $\int \frac{\sqrt{arctgx}}{1+x^2}dx$	6. $\int \frac{1}{x\sqrt{\ln^2 x+4}}dx$
7. $\int xctgx^2dx$	8. $\int \frac{e^x}{4+e^{2x}}dx$	9. $\int \frac{\arcsin^5 x}{\sqrt{1-x^2}}dx$

Variant 4

1. $\int \frac{\cos(\ln x)}{x}dx$	2. $\int \frac{\cos x}{\sqrt{1-\sin^2 x}}dx$	3. $\int \frac{\sqrt{2+ctgx}}{\sin^2 x}dx$
4. $\int \frac{x^2}{\sqrt[3]{x^3+2}}dx$	5. $\int \frac{\sqrt{arctg^3 x}}{1+x^2}dx$	6. $\int \frac{1}{x(\ln^2 x+4)}dx$
7. $\int x\cos x^2dx$	8. $\int \frac{e^x}{4-e^{2x}}dx$	9. $\int \frac{\sqrt{\arcsin x}}{\sqrt{1-x^2}}dx$

Variant 5

1. $\int \frac{\sqrt[3]{\ln x+1}}{x}dx$	2. $\int \frac{\sin x}{(1+\cos x)^2}dx$	3. $\int \frac{\sin 2x}{\sqrt{1+\cos 2x}}dx$
4. $\int \frac{x^3}{\sqrt{5+x^4}}dx$	5. $\int \frac{\sqrt[4]{arctg\,2x}}{1+4x^2}dx$	6. $\int \frac{1}{x(2\ln x+1)^2}dx$
7. $\int x^2\sin x^3dx$	8. $\int \frac{e^x}{1+2e^x}dx$	9. $\int x\left(\sqrt{3+x^2}\right)dx$

Variant 6

1. $\int \frac{\sin(\ln x)}{x}dx$	2. $\int \cos x\sin^3 xdx$	3. $\int \frac{dx}{\sqrt{tgx}\cos^2 x}$
4. $\int x^3\sqrt{x^4+4}dx$	5. $\int \frac{dx}{(1+x^2)arctg^3 x}$	6. $\int \frac{\ln^2 x+4}{x}dx$
7. $\int xctgx^2dx$	8. $\int \frac{e^{2x}}{16+e^{4x}}dx$	9. $\int \frac{\arcsin^4 x}{\sqrt{1-x^2}}dx$

Variant 7

1. $\int \frac{dx}{x(2\ln x+3)^3}$	2. $\int \frac{\cos 3x}{\sqrt{2+3\sin 3x}}dx$	3. $\int \frac{tg^2 x}{\sin^2 x}dx$

4. $\int \frac{x^3}{\sqrt[3]{2x^4+1}}dx$	5. $\int \frac{dx}{(1+x^2)arctgx}$	6. $\int \frac{\sqrt{\ln x+4}}{x}dx$
7. $\int xctgx^2dx$	8. $\int e^x(4+3e^x)^3dx$	9. $\int \frac{dx}{\arcsin x\sqrt{1-x^2}}$

Variant 8

1. $\int \frac{\ln^3 x}{x}dx$	2. $\int \frac{\sin x}{\sqrt{1+\cos^2 x}}dx$	3. $\int \frac{(ctgx+1)^2}{\sin^2 x}dx$
4. $\int x^3\cos x^4dx$	5. $\int \frac{\sqrt{arctgx}}{1+x^2}dx$	6. $\int \frac{1}{x(\ln^2 x+4)}dx$
7. $\int \frac{1}{x^2}\sin\frac{1}{x}dx$	8. $\int \frac{e^x}{\sqrt{9-e^{2x}}}dx$	9. $\int \frac{\sqrt{\arcsin x}}{\sqrt{1-x^2}}dx$

Variant 9

1. $\int \frac{\ln^7 x}{x}dx$	2. $\int \frac{\cos 2x}{(7+3\sin 2x)^2}dx$	3. $\int \frac{2^{tgx}}{\cos^2 x}dx$
4. $\int \frac{x^2}{\sqrt{2+2x^3}}dx$	5. $\int \frac{arctg^6 x}{1+x^2}dx$	6. $\int \frac{dx}{\arcsin x\sqrt{1-x^2}}$
7. $\int \frac{xdx}{4+x^4}$	8. $\int \frac{e^x}{\sqrt{5+2e^x}}dx$	9. $\int \frac{xdx}{5-x^2}$

Variant 10

1. $\int \frac{(2+5\ln x)^2}{x}dx$	2. $\int \cos x\sin^7 xdx$	3. $\int \frac{2^{tgx}}{\cos^2 x}dx$
4. $\int x\sin(2x^2+1)dx$	5. $\int \frac{dx}{(1+x^2)arctgx}$	6. $\int e^x\cos e^xdx$
7. $\int \frac{1}{x}\cos(\ln x)dx$	8. $\int \frac{e^x}{\sqrt{7+e^{2x}}}dx$	9. $\int \frac{\arcsin^4 x}{\sqrt{1-x^2}}dx$

Variant 11

1. $\int \frac{(2\ln x+3)^3}{x}dx$	2. $\int \cos x5^{\sin x}dx$	3. $\int \frac{ctg^2x}{\sin^2 x}dx$

4. $\int x^3 \sin(2-x^4)dx$	5. $\int \frac{e^{arcctgx}}{(1+x^2)}dx$	6. $\int \frac{\sqrt{\ln x+4}}{x}dx$
7. $\int x^4 tgx^5 dx$	8. $\int \frac{(4+ctgx)^2}{\sin^2 x}dx$	9. $\int \frac{\arccos^3 x}{\sqrt{1-x^2}}dx$

Variant 12

1. $\int \frac{\ln^5 x}{x}dx$	2. $\int \frac{\sin 2x}{\sqrt{1+3\cos 2x}}dx$	3. $\int \frac{\sqrt{3ctgx+7}}{\sin^2 x}dx$
4. $\int x^3 tgx^4 dx$	5. $\int \frac{3^{arctgx}}{1+x^2}dx$	6. $\int \frac{1}{x\sqrt{1-\ln^2 x}}dx$
7. $\int \frac{1}{x^2}\cos\frac{1}{x}dx$	8. $\int e^x \cos e^x dx$	9. $\int \frac{5^{\arccos x}}{\sqrt{1-x^2}}dx$

Variant 13

1. $\int \frac{\sqrt[5]{\ln x}}{x}dx$	2. $\int \frac{\cos 3x}{(2+\sin 3x)^3}dx$	3. $\int \frac{2^{tgx}}{\cos^2 x}dx$
4. $\int \frac{x^2}{x^3-5}dx$	5. $\int \frac{\sqrt{arctg^3 x}}{1+x^2}dx$	6. $\int \frac{7^{\arcsin x}}{\sqrt{1-x^2}}dx$
7. $\int \frac{xdx}{\sqrt{9-x^4}}$	8. $\int \frac{e^x}{\sqrt{9+e^x}}dx$	9. $\int \frac{xdx}{25-x^2}$

Variant 14

1. $\int \frac{1}{x\sqrt[3]{\ln x+2}}dx$	2. $\int \frac{\sin 2x}{(8-\cos 2x)}dx$	3. $\int \sin 2x e^{\cos 2x}dx$
4. $\int \frac{x}{\sqrt{15+x^2}}dx$	5. $\int \frac{arctg^2 x}{1+x^2}dx$	6. $\int \frac{1}{x(\ln x-1)^2}dx$
7. $\int x\cos x^2 dx$	8. $\int \frac{e^x}{4-e^{2x}}dx$	9. $\int x^2\sqrt{3-x^3}dx$

Variant 15

1. $\int \frac{1}{x\sqrt{2\ln x+9}}dx$	2. $\int \frac{\cos x}{(2-\sin x)^3}dx$	3. $\int \frac{5^{tgx}}{\cos^2 x}dx$

4. $\int x\sqrt[3]{x^2+4}dx$	5. $\int \frac{\sqrt{arctg^3 x}}{1+x^2}dx$	6. $\int \frac{\ln^4 x}{x}dx$
7. $\int xctgx^2 dx$	8. $\int \frac{e^x}{\sqrt{1+2e^{2x}}}dx$	9. $\int \frac{\arcsin^5 x}{\sqrt{1-x^2}}dx$

Variant 16

1. $\int \frac{\sqrt{2\ln x-1}}{x}dx$	2. $\int \frac{\cos 5x}{\sqrt{2+\sin 5x}}dx$	3. $\int \frac{\sqrt[3]{tgx}}{\cos^2 x}dx$
4. $\int \frac{x}{\sqrt[3]{5x^2+2}}dx$	5. $\int \frac{7^{arctgx}}{1+x^2}dx$	6. $\int \frac{1}{x\sqrt{9-\ln^2 x}}dx$
7. $\int x\cos(3x^2-2)dx$	8. $\int \frac{e^x}{16+e^{2x}}dx$	9. $\int \frac{\arccos^4 x}{\sqrt{1-x^2}}dx$

Variant 17

1. $\int \frac{\ln^9 x}{x}dx$	2. $\int \frac{\cos 6x}{\sqrt{1-\sin^2 6x}}dx$	3. $\int \frac{\sqrt[7]{4+3ctgx}}{\sin^2 x}dx$
4. $\int \frac{x^5}{\sqrt{x^6+9}}dx$	5. $\int \frac{\sqrt{arctg^5 x}}{1+x^2}dx$	6. $\int \frac{1}{x(\ln^2 x+9)}dx$
7. $\int x^4\cos x^5 dx$	8. $\int \frac{e^x}{3+e^{2x}}dx$	9. $\int \frac{\sqrt{\arcsin^3 x}}{\sqrt{1-x^2}}dx$

Variant 18

1. $\int \frac{\sqrt[8]{5\ln x+7}}{x}dx$	2. $\int \frac{\sin x}{(1+2\cos x)^5}dx$	3. $\int \frac{\sin 4x}{\sqrt{4+\cos 4x}}dx$
4. $\int \frac{x^2}{\sqrt{8+5x^3}}dx$	5. $\int \frac{\sqrt[3]{arctgx}}{1+x^2}dx$	6. $\int \frac{1}{x(2\ln x+1)^2}dx$
7. $\int x^3 5^{x^4}dx$	8. $\int \frac{e^x}{7-2e^x}dx$	9. $\int \frac{3^{ctgx}}{\sin^2 x}dx$

Variant 19

1. $\int \frac{\cos(2+5\ln x)}{x}dx$	2. $\int \cos^3 x\sin x dx$	3. $\int \frac{5^{tgx}}{\cos^2 x}dx$

4. $\int xe^{2x^2+1}dx$	5. $\int \frac{dx}{(1+x^2)arctg^4 x}$	6. $\int \frac{e^x}{\cos^2 e^x}dx$
7. $\int \frac{1}{x}\sin(\ln x)dx$	8. $\int \frac{e^x}{\sqrt{7-e^{2x}}}dx$	9. $\int \frac{\arcsin x}{\sqrt{1-x^2}}dx$

Variant 20

1. $\int \frac{(2-3\ln x)^3}{x}dx$	2. $\int \cos x 6^{\sin x}dx$	3. $\int \frac{ctg^2 x}{\sin^2 x}dx$
4. $\int x^3 \sin(5-3x^4)dx$	5. $\int \frac{e^{arctgx}}{(1+x^2)}dx$	6. $\int \frac{\sqrt{7\ln x+4}}{x}dx$
7. $\int xtg(2+x^2)dx$	8. $\int \frac{(1+tgx)^2}{\cos^2 x}dx$	9. $\int \frac{\arccos^6 x}{\sqrt{1-x^2}}dx$

§ 3. *Integration of Quadratic Form in Denominator*

Let us consider four types of integrals:

$$I_1 = \int \frac{dx}{X}; \quad I_2 = \int \frac{dx}{\sqrt{X}}; \; I_3 = \int \frac{A› + e}{X}dx; \quad I_4 = \int \frac{A› + e}{\sqrt{X}}dx,$$

where $X = ax^2 + bx + c$.

Integrals I_1 and I_2 are simplified to standard integrals (13-16 from the table of fundamental standard integrals) by completing the square. To complete the square, we can simply add and subtract term $\frac{b^2}{4a^2}$:

$$X = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right].$$

Example 1.

$$I = \int \frac{dx}{\sqrt{x^2-12x+11}} = \int \frac{dx}{\sqrt{x^2-12x+36-36+11}} = \int \frac{dx}{\sqrt{(x-6)^2-25}} =$$

$$= \ln\left|(x-6) + \sqrt{(x-6)^2-25}\right| + C$$

For integrals I_3, I_4 firstly we should create the differential of X in numerator:

$$I_3=\int\frac{Ax+B}{X}dx=\left\{\begin{array}{l}u=ax^2+bx+c\\du=(2ax+b)dx\end{array}\right\}=\int\frac{\frac{A}{2a}(2ax+b)+B-\frac{Ab}{2a}}{ax^2+bx+c}dx=$$

$$=\frac{A}{2a}\int\frac{2ax+b}{ax^2+bx+c}dx+\left(B-\frac{Ab}{2a}\right)\int\frac{dx}{ax^2+bx+c}=$$

$$=\frac{A}{2a}\int\frac{d\left(ax^2+bx+c\right)}{ax^2+bx+c}+\frac{1}{a}\left(B-\frac{Ab}{2a}\right)\int\frac{dx}{\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)}.$$

The first integral type is $\int\frac{du}{u}$ (integrand is a fraction whose numerator is the differential coefficient of the denominator), the second integral simplified to standard integrals (13-14) depending on the sign of the expression $\frac{c}{a}-\frac{b^2}{4a^2}$. Similarly, integral I_4 simplified to standard integrals (15-16).

Example 2.

$$I=\int\frac{x\,dx}{\sqrt{4x^2-12x+1}}=\left\{\begin{array}{l}u=4x^2-12x+1\\du=(8x-12)dx\end{array}\right\}=\int\frac{\frac{1}{8}(8x-12)+\frac{12}{8}}{\sqrt{4x^2-12x+1}}dx=$$

$$=\frac{1}{8}\int\frac{8x-12}{\sqrt{4x^2-12x+1}}dx+\frac{3}{2\cdot 2}\int\frac{dx}{\sqrt{x^2-3x+\frac{1}{4}}}=\frac{1}{8}\int\frac{du}{\sqrt{u}}+\frac{3}{4}\int\frac{d(x-3/2)}{\sqrt{(x-3/2)^2-2}}.$$

Both integrals are simplified to the tabular form. The first term corresponds to the formula (2a) and the second to (16) (substitution $u = x - 3/2$, $a^2 = 2$), given in the table of Fundamental Standard Integrals.

Therefore,

$$I=\frac{1}{8}\cdot 2\sqrt{u}+\frac{3}{4}\ln\left|x-\frac{3}{2}+\sqrt{\left(x-\frac{3}{2}\right)^2-2}\right|+C=$$

$$=\frac{1}{4}\sqrt{4x^2-12x+1}+\frac{3}{4}\ln\left|x-\frac{3}{2}+\sqrt{\left(x-\frac{3}{2}\right)^2-2}\right|+C.$$

Individual Task 3

Variant 1

1. $\int \frac{dx}{x^2-3x+5}$	2. $\int \frac{dx}{\sqrt{x^2-3x+5}}$	3. $\int \frac{(2x-1)dx}{\sqrt{x^2-3x+5}}$
4. $\int \frac{(3x+2)dx}{x^2-3x+5}$	5. $\int \frac{dx}{\sqrt{1+6x-x^2}}$	6. $\int \frac{dx}{3x^2-x+5}$
7. $\int \frac{(3x+2)dx}{3x^2-3x+5}$	8. $\int \frac{(7x-2)dx}{\sqrt{2x^2-3x+5}}$	9. $\int \frac{xdx}{\sqrt{x^2+4x-5}}$

Variant 2

1. $\int \frac{dx}{x^2+3x+1}$	2. $\int \frac{dx}{\sqrt{x^2+3x+1}}$	3. $\int \frac{(2x+7)dx}{\sqrt{x^2+3x+1}}$
4. $\int \frac{(5x+2)dx}{x^2+3x+1}$	5. $\int \frac{dx}{\sqrt{1+4x-x^2}}$	6. $\int \frac{dx}{2x^2+x+1}$
7. $\int \frac{(x+2)dx}{3x^2+3x+1}$	8. $\int \frac{(x-2)dx}{\sqrt{2x^2+3x+1}}$	9. $\int \frac{(x+3)dx}{\sqrt{x^2+12x+11}}$

Variant 3

1. $\int \frac{dx}{x^2+4x+1}$	2. $\int \frac{dx}{\sqrt{x^2+4x+1}}$	3. $\int \frac{(2x+5)dx}{\sqrt{x^2+4x+1}}$
4. $\int \frac{(5x+2)dx}{x^2+4x+1}$	5. $\int \frac{dx}{\sqrt{1+4x-x^2}}$	6. $\int \frac{dx}{2x^2+x+1}$
7. $\int \frac{(x+2)dx}{2x^2+4x+1}$	8. $\int \frac{(x+2)dx}{\sqrt{2x^2+4x+1}}$	9. $\int \frac{(x-3)dx}{\sqrt{x^2+6x+11}}$

Variant 4

1. $\int \frac{dx}{x^2+6x+1}$	2. $\int \frac{dx}{\sqrt{x^2+6x+1}}$	3. $\int \frac{(2x+1)dx}{\sqrt{x^2+6x+1}}$
4. $\int \frac{(x+2)dx}{x^2+6x+1}$	5. $\int \frac{dx}{\sqrt{1+4x-x^2}}$	6. $\int \frac{dx}{2x^2+x+1}$
7. $\int \frac{(x+1)dx}{2x^2+3x+1}$	8. $\int \frac{(x+2)dx}{\sqrt{2x^2+3x+1}}$	9. $\int \frac{(x-3)dx}{\sqrt{x^2+8x+1}}$

Variant 5

1. $\int \frac{dx}{x^2+8x+1}$	2. $\int \frac{dx}{\sqrt{x^2+8x+1}}$	3. $\int \frac{(2x+1)dx}{\sqrt{x^2+8x+1}}$
4. $\int \frac{xdx}{x^2+8x+1}$	5. $\int \frac{dx}{\sqrt{2+4x-x^2}}$	6. $\int \frac{dx}{2x^2+2x+1}$
7. $\int \frac{(x+9)dx}{2x^2+2x+1}$	8. $\int \frac{(x+2)dx}{\sqrt{5x^2+3x-1}}$	9. $\int \frac{(x+3)dx}{\sqrt{x^2+2x-1}}$

Variant 6

1. $\int \frac{dx}{x^2+2x+5}$	2. $\int \frac{dx}{\sqrt{x^2+2x+5}}$	3. $\int \frac{(2x+1)dx}{\sqrt{x^2+2x+5}}$
4. $\int \frac{xdx}{x^2+2x+5}$	5. $\int \frac{dx}{\sqrt{2-4x-x^2}}$	6. $\int \frac{dx}{2x^2+x+2}$
7. $\int \frac{(x-1)dx}{2x^2+x+2}$	8. $\int \frac{(x+2)dx}{\sqrt{3x^2+x-2}}$	9. $\int \frac{(x+3)dx}{\sqrt{3x^2+x-2}}$

Variant 7

1. $\int \frac{dx}{x^2+4x-9}$	2. $\int \frac{dx}{\sqrt{x^2+4x-9}}$	3. $\int \frac{(2x-1)dx}{\sqrt{x^2+4x-9}}$
4. $\int \frac{xdx}{x^2+4x-9}$	5. $\int \frac{dx}{\sqrt{7+2x-x^2}}$	6. $\int \frac{dx}{2x^2+5x-1}$
7. $\int \frac{(x-1)dx}{2x^2+5x-1}$	8. $\int \frac{(x+1)dx}{\sqrt{x^2+7x-2}}$	9. $\int \frac{(3x+4)dx}{\sqrt{x^2+7x-2}}$

Variant 8

1. $\int \frac{dx}{x^2-2x-5}$	2. $\int \frac{dx}{\sqrt{x^2-2x-5}}$	3. $\int \frac{(2x-1)dx}{\sqrt{x^2-2x-5}}$
4. $\int \frac{xdx}{x^2-2x-5}$	5. $\int \frac{dx}{\sqrt{1+2x-x^2}}$	6. $\int \frac{dx}{2x^2-5x-1}$
7. $\int \frac{(x-1)dx}{2x^2-5x-1}$	8. $\int \frac{(x+1)dx}{\sqrt{x^2+3x+4}}$	9. $\int \frac{(3x+2)dx}{\sqrt{x^2+3x+4}}$

Variant 9

1. $\int \frac{dx}{x^2-2x-1}$	2. $\int \frac{dx}{\sqrt{x^2-2x-1}}$	3. $\int \frac{(2x-1)dx}{\sqrt{x^2-2x-1}}$

4. $\int \frac{xdx}{x^2-2x-1}$	5. $\int \frac{dx}{\sqrt{5-2x-x^2}}$	6. $\int \frac{dx}{2x^2-x-1}$
7. $\int \frac{(x-1)dx}{2x^2-x-1}$	8. $\int \frac{(x+1)dx}{\sqrt{x^2+x+4}}$	9. $\int \frac{(3x+2)dx}{\sqrt{x^2+x+4}}$

Variant 10

1. $\int \frac{dx}{x^2-2x-4}$	2. $\int \frac{dx}{\sqrt{x^2-2x-4}}$	3. $\int \frac{(2x-1)dx}{\sqrt{x^2-2x-4}}$
4. $\int \frac{xdx}{x^2-2x-4}$	5. $\int \frac{dx}{\sqrt{8-4x-x^2}}$	6. $\int \frac{dx}{2x^2-2x+7}$
7. $\int \frac{(x-1)dx}{2x^2-2x+7}$	8. $\int \frac{(x+1)dx}{\sqrt{x^2+x+9}}$	9. $\int \frac{(3x+2)dx}{\sqrt{x^2+x+9}}$

Variant 11

1. $\int \frac{dx}{x^2-2x-3}$	2. $\int \frac{dx}{\sqrt{x^2-2x-3}}$	3. $\int \frac{(4x-1)dx}{\sqrt{x^2-2x-3}}$
4. $\int \frac{xdx}{x^2-2x-3}$	5. $\int \frac{dx}{\sqrt{1-6x-x^2}}$	6. $\int \frac{dx}{4x^2-2x+1}$
7. $\int \frac{(x-1)dx}{4x^2-2x+1}$	8. $\int \frac{(x+1)dx}{\sqrt{2x^2+x+2}}$	9. $\int \frac{(x-2)dx}{\sqrt{2x^2+x+2}}$

Variant 12

1. $\int \frac{dx}{x^2-2x+5}$	2. $\int \frac{dx}{\sqrt{x^2-2x+5}}$	3. $\int \frac{(6x-1)dx}{\sqrt{x^2-2x+5}}$
4. $\int \frac{xdx}{x^2-2x+5}$	5. $\int \frac{dx}{\sqrt{7-2x-x^2}}$	6. $\int \frac{dx}{3x^2-2x+1}$
7. $\int \frac{(x-1)dx}{3x^2-2x+1}$	8. $\int \frac{(x+1)dx}{\sqrt{4x^2+x+2}}$	9. $\int \frac{(x-2)dx}{\sqrt{4x^2+x+2}}$

Variant 13

1. $\int \frac{dx}{x^2-2x+12}$	2. $\int \frac{dx}{\sqrt{x^2-2x+12}}$	3. $\int \frac{(6x-1)dx}{\sqrt{x^2-2x+12}}$
4. $\int \frac{xdx}{x^2-2x+12}$	5. $\int \frac{dx}{\sqrt{6-2x-x^2}}$	6. $\int \frac{dx}{2x^2-x+4}$
7. $\int \frac{(x-1)dx}{2x^2-x+4}$	8. $\int \frac{(x+1)dx}{\sqrt{9x^2+x+2}}$	9. $\int \frac{(x-2)dx}{\sqrt{9x^2+x+2}}$

Variant 14

1. $\int \frac{dx}{x^2+2x+12}$	2. $\int \frac{dx}{\sqrt{x^2+2x+12}}$	3. $\int \frac{(6x-1)dx}{\sqrt{x^2+2x+12}}$
4. $\int \frac{xdx}{x^2+2x+12}$	5. $\int \frac{dx}{\sqrt{6+2x-x^2}}$	6. $\int \frac{dx}{2x^2-3x-4}$
7. $\int \frac{(x-1)dx}{2x^2-3x-4}$	8. $\int \frac{(x+1)dx}{\sqrt{9x^2-x+2}}$	9. $\int \frac{(x-2)dx}{\sqrt{9x^2-x+2}}$

Variant 15

1. $\int \frac{dx}{x^2-2x+11}$	2. $\int \frac{dx}{\sqrt{x^2-2x+11}}$	3. $\int \frac{(6x-1)dx}{\sqrt{x^2-2x+11}}$
4. $\int \frac{xdx}{x^2-2x+11}$	5. $\int \frac{dx}{\sqrt{1+4x-x^2}}$	6. $\int \frac{dx}{2x^2-x-4}$
7. $\int \frac{(3x-1)dx}{2x^2-x-4}$	8. $\int \frac{(5x+1)dx}{\sqrt{9x^2-x+3}}$	9. $\int \frac{(x-2)dx}{\sqrt{9x^2-x+3}}$

Variant 16

1. $\int \frac{dx}{x^2-2x+10}$	2. $\int \frac{dx}{\sqrt{x^2-2x+10}}$	3. $\int \frac{(6x-1)dx}{\sqrt{x^2-2x+10}}$
4. $\int \frac{xdx}{x^2-2x+10}$	5. $\int \frac{dx}{\sqrt{9+2x-x^2}}$	6. $\int \frac{dx}{2x^2-2x-3}$
7. $\int \frac{(3x-1)dx}{2x^2-2x-3}$	8. $\int \frac{(5x+1)dx}{\sqrt{x^2-x+8}}$	9. $\int \frac{(x-2)dx}{\sqrt{x^2-x+8}}$

Variant 17

1. $\int \frac{dx}{x^2-3x+1}$	2. $\int \frac{dx}{\sqrt{x^2-3x+1}}$	3. $\int \frac{(2x-1)dx}{\sqrt{x^2-3x+1}}$
4. $\int \frac{(3x+2)dx}{x^2-3x+1}$	5. $\int \frac{dx}{\sqrt{1+x-x^2}}$	6. $\int \frac{dx}{3x^2-2x+5}$
7. $\int \frac{(3x+2)dx}{3x^2-2x+5}$	8. $\int \frac{(x-2)dx}{\sqrt{2x^2-3x+5}}$	9. $\int \frac{xdx}{\sqrt{x^2+6x-5}}$

Variant 18

1. $\int \frac{dx}{x^2-x+1}$	2. $\int \frac{dx}{\sqrt{x^2-x+1}}$	3. $\int \frac{(2x-1)dx}{\sqrt{x^2-x+1}}$

4. $\int \frac{(x+2)dx}{x^2-x+1}$	5. $\int \frac{dx}{\sqrt{5+x-x^2}}$	6. $\int \frac{dx}{2x^2-2x+5}$
7. $\int \frac{(3x+2)dx}{2x^2-2x+5}$	8. $\int \frac{(x-2)dx}{\sqrt{x^2-3x+5}}$	9. $\int \frac{xdx}{\sqrt{4x^2+6x-5}}$

Variant 19

1. $\int \frac{dx}{x^2-2x+3}$	2. $\int \frac{dx}{\sqrt{x^2-2x+3}}$	3. $\int \frac{(2x-1)dx}{\sqrt{x^2-2x+3}}$
4. $\int \frac{(x+2)dx}{x^2-2x+3}$	5. $\int \frac{dx}{\sqrt{5-x-x^2}}$	6. $\int \frac{dx}{2x^2-4x+1}$
7. $\int \frac{(3x+2)dx}{2x^2-4x+1}$	8. $\int \frac{(x-2)dx}{\sqrt{4x^2-3x+5}}$	9. $\int \frac{xdx}{\sqrt{9x^2+6x-5}}$

Variant 20

1. $\int \frac{dx}{x^2-6x+3}$	2. $\int \frac{dx}{\sqrt{x^2-6x+3}}$	3. $\int \frac{(2x-1)dx}{\sqrt{x^2-6x+3}}$
4. $\int \frac{(x+2)dx}{x^2-6x+3}$	5. $\int \frac{dx}{\sqrt{1+x-4x^2}}$	6. $\int \frac{dx}{2x^2-8x+5}$
7. $\int \frac{(3x+2)dx}{2x^2-8x+5}$	8. $\int \frac{(x-2)dx}{\sqrt{16x^2-3x+5}}$	9. $\int \frac{xdx}{\sqrt{9x^2+6x+5}}$

§ 4. *Integration of Some Trigonometric Functions*

4.1. Let us consider integral of the type:

$$\int \sin^m x \cos^n x\, dx . \qquad (10)$$

a) If at least one of the parameters *m* or *n* is odd positive number then integration is reduced to search a primitive of power function.

Indeed, if $m = 2k + 1$, $k \in N$, then $\sin^m x$ can be transformed into a product of $\sin^{m-1} x \cdot \sin x$, where ($m - 1 = 2k$) is an even positive number. The first factor ($\sin^{m-1} x$) should be transformed into *cos* function, using $\sin^2 x = 1 - \cos^2 x$. The second one will be the part of the differential *du* while substitution $u = \cos x$:

$$\int \sin^{2k+1} x \cos^n x\, dx = \int \sin^{2k} x \cos^n x \sin x\, dx =$$

$$= -\int (1-\cos^2 x)^k \cos^n x\, d(\cos x) = \{u = \cos x\} = -\int \left(1-u^2\right)^k u^n du .$$

Now, open the brackets and we get the $k+1$ terms of standard forms.

Example 1.

$$\int \sqrt[3]{\sin x}\cos^5 x\,dx = \int \sqrt[3]{\sin x}\cos^4 x\cos x\,dx =$$

$$= \int \sqrt[3]{\sin x}\left(1-\sin^2 x\right)^2 d\left(\sin x\right) = \left\{u=\sin x\right\} = \int u^{1/3}\left(1-u^2\right)^2 du =$$

$$= \int\left(u^{1/3} - 2u^{7/3} + u^{13/3}\right)du = \frac{u^{4/3}}{4/3} - 2\frac{u^{10/3}}{10/3} + \frac{u^{16/3}}{16/3} + C =$$

$$= \frac{3}{4}\sqrt[3]{\sin^4 x} - \frac{3}{5}\sqrt[3]{\sin^{10} x} + \frac{3}{16}\sqrt[3]{\sin^{16} x} + C.$$

b) If both powers m and n are even positive numbers (one of which may be zero), the integral can be simplified using formulas

$$\sin^2\alpha = \frac{1}{2}\left(1-\cos 2\alpha\right),$$

$$\cos^2\alpha = \frac{1}{2}\left(1+\cos 2\alpha\right), \qquad (11)$$

$$\sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha.$$

Example 2.

$$I = \int \sin^2 x\cos^4 x\,dx = \int \sin^2 x\cos^2 x\cos^2 x\,dx =$$

$$= \int \frac{1}{4}\sin^2 2x\cdot\frac{1}{2}\left(1+\cos 2x\right)dx = \frac{1}{8}\int \sin^2 2x\,dx + \frac{1}{8}\int \sin^2 2x\cos 2x\,dx.$$

The first integral could be simplified using formulas (11) one more time. The second one is the integral of the type a). So,

$$I = \frac{1}{8}\cdot\frac{1}{2}\int\left(1-\cos 4x\right)dx + \frac{1}{8}\cdot\frac{1}{2}\int \sin^2 2x\,d\left(\sin 2x\right) = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x\right) + \frac{1}{48}\sin^3 2x + C.$$

4.2. Now, let us consider integral of the type:

$$\int tg^m x\cdot\sec^n x\,dx,\quad \int ctg^m x\cdot\operatorname{cosec}^n x\,dx. \qquad (12)$$

If the index n is an even positive number, then for any value of m we could use the trigonometric relations:

$$1 + tg^2x = sec^2x, \qquad 1 + ctg^2x = cosec^2x, \qquad (13)$$

Let $u=tgx$ or $u=ctgx$, then $du = \dfrac{dx}{\cos^2 x} = \sec^2 x dx$ or $du = -\dfrac{dx}{\sin^2 x} = -\operatorname{cosec}^2 x dx$.

Example 3.

$$\int \frac{\sec^4 x}{\sqrt{tgx}}dx = \int \frac{\sec^2 x}{\sqrt{tgx}}\sec^2 x\,dx = \int \frac{1+tg^2 x}{\sqrt{tgx}}d\left(tgx\right) = \left\{u = tgx\right\} =$$

$$= \int \frac{1+u^2}{\sqrt{u}}du = \int \frac{du}{\sqrt{u}} + \int u^{3/2}du = 2\sqrt{u} + \frac{u^{5/2}}{5/2} + C = 2\sqrt{tgx} + \frac{2}{5}\sqrt{tg^5 x} + C.$$

4.3. Integrals of the product of sine and cosine functions in the 1st power.

$$\int \sin mx \cos nx\,dx,\ \int \sin mx \sin nx\,dx,\ \int \cos mx \cos nx\,dx. \qquad (14)$$

Such products could be transformed into sum using formulas:

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right],$$
$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right], \qquad (15)$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right].$$

Example 4.

$$\int \sin 7x \sin 3x\,dx = \frac{1}{2}\int(\cos 4x - \cos 10x)dx = \frac{1}{2\cdot 4}\int \cos 4x\,d(4x) -$$
$$-\frac{1}{2\cdot 10}\int \cos 10x\,d(10x) = \frac{1}{8}\sin 4x - \frac{1}{20}\sin 10x + C.$$

Individual Task 4

Variant 1

1. $\int \sin^2 x \cos^3 x\,dx$	2. $\int \sin^5 x \cos^4 x\,dx$	3. $\int \sin^5 x \cos^3 x\,dx$
4. $\int \sin^2 x \cos^2 x\,dx$	5. $\int \frac{\sec^4 x}{tg^3 x}dx$	6. $\int \frac{ctg^4 x}{\sin^2 x}dx$
7. $\int \sin 5x \sin 3x\,dx$	8. $\int \cos 5x \sin 3x\,dx$	9. $\int \cos 5x \cos 3x\,dx$

Variant 2

1. $\int \sin^3 x \cos^2 x\,dx$	2. $\int \sin^4 x \cos^5 x\,dx$	3. $\int \sin^7 x \cos^3 x\,dx$
4. $\int \sin^4 x \cos^2 x\,dx$	5. $\int \frac{\sqrt{tgx}dx}{\cos^2 x}$	6. $\int \frac{ctg^2 x}{\sin^4 x}dx$
7. $\int \sin 6x \sin 3x\,dx$	8. $\int \cos 7x \sin 3x\,dx$	9. $\int \cos 5x \cos 2x\,dx$

Variant 3

1. $\int \sin^7 x \cos^4 x\,dx$	2. $\int \sin^6 x \cos^5 x\,dx$	3. $\int \sin^7 x \cos^{13} x\,dx$
4. $I = \int \sin^2 x \cos^4 x\,dx$	5. $\int \frac{dx}{\cos^6 x}$	6. $\int \frac{ctgx}{\sin^4 x}dx$
7. $\int \sin 3x \sin 2x\,dx$	8. $\int \cos 7x \sin 5x\,dx$	9. $\int \cos x \cos 7x\,dx$

Variant 4

1. $\int \sin^3 x\cos^8 x\,dx$	2. $\int \sin^6 x\cos^3 x\,dx$	3. $\int \sin^3 x\cos^3 x\,dx$
4. $\int \sin^2 x\cos^2 x\,dx$	5. $\int \frac{dx}{\cos^4 x}$	6. $\int \frac{\sqrt{ctgx}}{\sin^4 x}dx$
7. $\int \sin 7x\sin 2x\,dx$	8. $\int \cos 2x\sin 5x\,dx$	9. $\int \cos x\cos 2x\,dx$

Variant 5

1. $\int \sin\ x\cos^6 x\,dx$	2. $\int \sin^4 x\cos^3 x\,dx$	3. $\int \sin^5 x\cos^3 x\,dx$
4. $\int \sin^4 x\,dx$	5. $\int \frac{\sqrt{tgx}dx}{\cos^4 x}$	6. $\int \frac{dx}{ctgx\sin^4 x}$
7. $\int \sin x\sin 2x\,dx$	8. $\int \cos x\sin 5x\,dx$	9. $\int \cos 4x\cos 2x\,dx$

Variant 6

1. $\int \sin^7 x\cos^2 x\,dx$	2. $\int \sin^4 x\cos^5 x\,dx$	3. $\int \sin^{15} x\cos^3 x\,dx$
4. $\int \cos^4 x\,dx$	5. $\int \frac{\sqrt{tgx}dx}{\cos^2 x}$	6. $\int \frac{dx}{\sqrt{ctgx}\sin^4 x}$
7. $\int \sin 8x\sin 2x\,dx$	8. $\int \cos 5x\sin 2x\,dx$	9. $\int \cos 4x\cos 6x\,dx$

Variant 7

1. $\int \sin^3 x\cos^4 x\,dx$	2. $\int \sin^2 x\cos^7 x\,dx$	3. $\int \sin^{15} x\cos x\,dx$
4. $\int \cos^2 x\,dx$	5. $\int \frac{dx}{tgx\cos^2 x}$	6. $\int \frac{dx}{\sqrt{ctgx}\sin^2 x}$
7. $\int \sin 2x\sin 2x\,dx$	8. $\int \cos 5x\sin 3x\,dx$	9. $\int \cos 4x\cos 2x\,dx$

Variant 8

1. $\int \sin^5 x\cos^2 x\,dx$	2. $\int \sin^4 x\cos^3 x\,dx$	3. $\int \sin^{11} x\cos^3 x\,dx$
4. $\int \sin^2 x\,dx$	5. $\int \frac{tg^2 xdx}{\cos^2 x}$	6. $\int \frac{\sqrt[3]{ctg^2 xdx}}{\sin^2 x}$
7. $\int \sin 7x\sin 2x\,dx$	8. $\int \cos 5x\sin x\,dx$	9. $\int \cos x\cos 2x\,dx$

Variant 9

1. $\int \sin^7 x\cos^2 x\,dx$	2. $\int \sin^6 x\cos^3 x\,dx$	3. $\int \sin^9 x\cos^3 x\,dx$
4. $\int \cos^4 x\sin^2 x\,dx$	5. $\int \frac{\sqrt[3]{tg^2 x}dx}{\cos^4 x}$	6. $\int \frac{\sqrt[3]{ctg^2 x}dx}{\sin^4 x}$
7. $\int \sin x\sin 2x\,dx$	8. $\int \cos x\sin 4x\,dx$	9. $\int \cos 7x\cos 2x\,dx$

Variant 10

1. $\int \sin^3 x \cos^4 x\,dx$	2. $\int \sin^4 x \cos^3 x\,dx$	3. $\int \sin^{19} x \cos^3 x\,dx$
4. $\int \cos^2 x \sin^4 x\,dx$	5. $\int \frac{tg^4 x dx}{\cos^4 x}$	6. $\int \frac{\sqrt[5]{ctg^4 x}dx}{\sin^4 x}$
7. $\int \sin 3x \sin x\,dx$	8. $\int \cos 7x \sin 4x\,dx$	9. $\int \cos 8x \cos 2x\,dx$

Variant 11

1. $\int \sin^3 x \cos^2 x\,dx$	2. $\int \sin^2 x \cos^5 x\,dx$	3. $\int \sin x \cos^3 x\,dx$
4. $\int \sin^6 x\,dx$	5. $\int \frac{\sqrt[5]{tg^4 x}dx}{\cos^4 x}$	6. $\int \frac{\sqrt[3]{ctg^2 x}dx}{\sin^4 x}$
7. $\int \sin 3x \sin 3x\,dx$	8. $\int \cos 2x \sin 4x\,dx$	9. $\int \cos 3x \cos 5x\,dx$

Variant 12

1. $\int \sin^4 x \cos^3 x\,dx$	2. $\int \sin^5 x \cos^2 x\,dx$	3. $\int \sin^5 x \cos^7 x\,dx$
4. $\int \sin^4 x\,dx$	5. $\int \frac{\sec^4 x}{tgx} dx$	6. $\int \frac{ctg^4 x}{\sin^4 x} dx$
7. $\int \sin 5x \sin 13x\,dx$	8. $\int \cos 5x \sin 2x\,dx$	9. $\int \cos 5x \cos 10x\,dx$

Variant 13

1. $\int \sin^3 x \cos^{12} x\,dx$	2. $\int \sin^{14} x \cos^5 x\,dx$	3. $\int \sin^7 x \cos^{31} x\,dx$
4. $\int \sin^6 x dx$	5. $\int \frac{\sqrt{tgx}dx}{\cos^4 x}$	6. $\int \frac{\sqrt{ctgx}}{\sin^4 x} dx$
7. $\int \sin 16x \sin 3x\,dx$	8. $\int \cos 7x \sin x\,dx$	9. $\int \cos 5x \cos 12x dx$

Variant 14

1. $\int \sin^3 x \cos^{14} x\,dx$	2. $\int \sin^6 x \cos^3 x\,dx$	3. $\int \sin^3 x \cos^{13} x\,dx$
4. $\int \cos^4 x\,dx$	5. $\int \frac{dx}{\cos^6 x}$	6. $\int \frac{\sqrt[5]{ctg^2 x}}{\sin^4 x} dx$
7. $\int \sin 3x \sin 12x\,dx$	8. $\int \cos x \sin 5x\,dx$	9. $\int \cos x \cos 4x\,dx$

Variant 15

1. $\int \sin^5 x \cos^8 x\,dx$	2. $\int \sin^6 x \cos^5 x\,dx$	3. $\int \sin^3 x \cos^{13} x\,dx$
4. $\int \sin^2 x \cos^2 x\,dx$	5. $\int \frac{tg^4 x dx}{\cos^4 x}$	6. $\int \frac{dx}{ctg^4 x \sin^4 x}$
7. $\int \sin 7x \sin 3x\,dx$	8. $\int \cos 2x \sin 6x\,dx$	9. $\int \cos 7x \cos 2x\,dx$

Variant 16

1. $\int \sin^3 x \cos^8 x\,dx$	2. $\int \sin^4 x \cos^5 x\,dx$	3. $\int \sin^5 x \cos^7 x\,dx$
4. $\int \cos^4 x\,dx$	5. $\int \frac{dx}{tg^2 x \cos^4 x}$	6. $\int \frac{dx}{ctg^2 x \sin^4 x}$
7. $\int \sin x \sin 12x\,dx$	8. $\int \cos 5x \sin 5x\,dx$	9. $\int \cos 4x \cos 8x\,dx$

Variant 17

1. $\int \sin^3 x \cos^4 x\,dx$	2. $\int \sin^4 x \cos x\,dx$	3. $\int \sin x \cos^{13} x\,dx$
4. $\int \cos^2 x\,dx$	5. $\int \frac{\sqrt{tgx}dx}{\cos^4 x}$	6. $\int \frac{dx}{\sqrt{ctgx} \sin^4 x}$
7. $\int \sin 4x \sin 2x\,dx$	8. $\int \cos 3x \sin 2x\,dx$	9. $\int \cos 4x \cos 2x\,dx$

Variant 18

1. $\int \sin^3 x \cos^8 x\,dx$	2. $\int \sin^6 x \cos^5 x\,dx$	3. $\int \sin^{15} x \cos x\,dx$
4. $\int \cos^2 x \sin^2 x\,dx$	5. $\int \frac{dx}{tgx \cos^4 x}$	6. $\int \frac{dx}{\sqrt{ctgx} \sin^4 x}$
7. $\int \sin 2x \sin 12x\,dx$	8. $\int \cos 5x \sin x\,dx$	9. $\int \cos 4x \cos 2x\,dx$

Variant 19

1. $\int \sin^5 x \cos^8 x\,dx$	2. $\int \sin^6 x \cos^3 x\,dx$	3. $\int \sin^9 x \cos^3 x\,dx$
4. $\int \sin^2 x\,dx$	5. $\int \frac{tg^2 x dx}{\cos^4 x}$	6. $\int \frac{\sqrt[3]{ctg^2 x} dx}{\sin^4 x}$
7. $\int \sin x \sin 2x\,dx$	8. $\int \cos 5x \sin 4x\,dx$	9. $\int \cos 7x \cos 2x\,dx$

Variant 20

1. $\int \sin^3 x \cos^4 x\,dx$	2. $\int \sin^8 x \cos^3 x\,dx$	3. $\int \sin^{19} x \cos^3 x\,dx$

4. $\int \cos^4 x\,dx$	5. $\int \frac{\sqrt[5]{tg^2 x}dx}{\cos^4 x}$	6. $\int \frac{ctg^2 x dx}{\sin^4 x}$
7. $\int \sin 9x \sin 2x\,dx$	8. $\int \cos 5x \sin 4x\,dx$	9. $\int \cos 7x \cos x\,dx$

§ 5. *Integration by Parts: Product of Two Functions*

Let $u = u(x)$ and $v = v(x)$ are continuous and differentiable functions. Directly from the product rule for differentiation $(uv)' = u'v + uv'$ follows: $uv' = (uv)' - u'v$.

Now we integrate this equation:

$$\int uv'\,dx = uv - \int vu'\,dx .$$

As we remember $u'dx = du$, $v'dx = dv$. Hence the formula of integration by parts is

$$\int u\,dv = uv - \int v\,du . \qquad (16)$$

Example 1.

$$\int x\cos 2x\,dx = \left\{ \begin{array}{c} u = x \\ dv = \cos 2x\,dx \end{array} \middle| \begin{array}{c} du = dx \\ v = \int \cos 2x\,dx = \frac{1}{2}\sin 2x \end{array} \right\} =$$

$$= \frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x\,dx = \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C .$$

In some cases the formula (8) should be used again several times.

Example 2.

$$\int x^2 e^{-x}\,dx = \left\{ \begin{array}{c} u = x^2 \\ dv = e^{-x}\,dx \end{array} \middle| \begin{array}{c} du = 2xdx \\ v = \int e^{-x}\,dx = -e^{-x} \end{array} \right\} =$$

$$= -x^2 e^{-x} + 2\int xe^{-x}\,dx = \left\{ \begin{array}{c} u = x \\ dv = e^{-x}\,dx \end{array} \middle| \begin{array}{c} du = dx \\ v = \int e^{-x}\,dx = -e^{-x} \end{array} \right\} = -x^2 e^{-x} +$$

$$+2\left(-xe^{-x} + \int e^{-x}\,dx\right) = -x^2 e^{-x} + 2\left(-xe^{-x} - e^{-x}\right) + C = -\left(x^2 + 2x + 2\right)e^{-x} + C$$

Integration by Parts formula (16) could be helpful even for cases with no simplification. For example:

Example 3.

$$I = \int \sqrt{x^2 + a^2}\,dx = \left\{ \begin{array}{c} u = \sqrt{x^2 + a^2} \\ dv = dx \end{array} \middle| \begin{array}{c} du = \frac{x}{\sqrt{x^2 + a^2}}dx \\ v = \int dx = x \end{array} \right\} =$$

$$= x\sqrt{x^2 + a^2} - \int \frac{x^2}{\sqrt{x^2 + a^2}}dx = x\sqrt{x^2 + a^2} - \int \frac{\left(x^2 + a^2\right) - a^2}{\sqrt{x^2 + a^2}}dx = x\sqrt{x^2 + a^2} -$$

$$-\int \frac{x^2+a^2}{\sqrt{x^2+a^2}}dx + a^2\int \frac{dx}{\sqrt{x^2+a^2}} = x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\,dx + a^2 \ln\left|x+\sqrt{x^2+a^2}\right|$$

We can write $I = x\sqrt{x^2+a^2} - I + a^2 \ln\left|x+\sqrt{x^2+a^2}\right|$ or

$$2I = x\sqrt{x^2+a^2} + a^2 \ln\left|x+\sqrt{x^2+a^2}\right|.$$

Hence

$$I = \int \sqrt{x^2+a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2+a^2} + a^2 \ln\left|x+\sqrt{x^2+a^2}\right|\right] + C. \qquad (17)$$

Individual Task 5

Variant 1

1. $\int (x+2)\sin 2x\,dx$	2. $\int \arcsin 2x\,dx$	3. $\int \ln x\,dx$
4. $\int (2x-5)e^{2x}\,dx$	5. $\int \frac{x}{\cos^2 x}dx$	6. $\int \cos x\,e^{2x}dx$
7. $\int (2x^2-5)\cos x\,dx$	8. $\int x arctgx\,dx$	9. $\int \ln^2 x\,dx$

Variant 2

1. $\int (2-x)\sin 3x\,dx$	2. $\int \arccos 2x\,dx$	3. $\int x\ln x\,dx$
4. $\int (3x+5)e^{3x}\,dx$	5. $\int \frac{x}{\sin^2 x}dx$	6. $\int \sin 3x\,e^{2x}dx$
7. $\int (x^2-x)\cos x\,dx$	8. $\int arctgx\,dx$	9. $\int \sin(\ln x)dx$

Variant 3

1. $\int (3-2x)\sin x\,dx$	2. $\int \arccos 3x\,dx$	3. $\int \frac{dx}{\cos^3 x}$
4. $\int (3x+1)e^{-x}\,dx$	5. $\int \ln(2x+3)\,dx$	6. $\int \sin x\,e^{-2x}dx$
7. $\int (3x^2-1)\cos 2x\,dx$	8. $\int x arctgx\,dx$	9. $\int \cos(\ln x)dx$

Variant 4

1. $\int x\sin(2x-1)\,dx$	2. $\int x\arccos x\,dx$	3. $\int \sqrt{x^2+1}dx$
4. $\int (x+1)e^{3-x}\,dx$	5. $\int \ln(1-2x)\,dx$	6. $\int \cos 5x\,e^{-x}dx$
7. $\int (x^2+1)\cos 3x\,dx$	8. $\int arctg\,2x\,dx$	9. $\int \sin(\ln 2x)dx$

Variant 5

1. $\int(1-2x)\sin 3x\,dx$	2. $\int x\arcsin x\,dx$	3. $\int\frac{dx}{\cos^3 x}$
4. $\int(7x+1)e^{4x}\,dx$	5. $\int\ln(4-3x)\,dx$	6. $\int\cos x\,e^{-2x}dx$
7. $\int(5-x^2)\cos x\,dx$	8. $\int arcctg\,3x\,dx$	9. $\int\cos(\ln\frac{x}{3})dx$

Variant 6

1. $\int(2-x)\sin 2x\,dx$	2. $\int\arcsin 4x\,dx$	3. $\int x^2\ln x\,dx$
4. $\int(3x-5)e^{2x}\,dx$	5. $\int\frac{x}{\cos^2 3x}dx$	6. $\int\cos 5x\,e^{x}dx$
7. $\int x^2\cos 2x\,dx$	8. $\int arctgx\,dx$	9. $\int\ln^2 x\,dx$

Variant 7

1. $\int(x+3)\sin x\,dx$	2. $\int\arccos x\,dx$	3. $\int\sqrt{x}\ln x\,dx$
4. $\int(x+5)e^{-3x}\,dx$	5. $\int\frac{x}{\sin^2 2x}dx$	6. $\int\sin x\,e^{-2x}dx$
7. $\int(4-x^2)\cos 2x\,dx$	8. $\int xarctgx\,dx$	9. $\int\sin(\ln 9x)dx$

Variant 8

1. $\int(3-2x)\cos x\,dx$	2. $\int x\arccos 2x\,dx$	3. $\int\frac{dx}{\cos^3 2x}$
4. $\int(x+1)e^{-2x}\,dx$	5. $\int\ln(2-x)\,dx$	6. $\int\sin 5x\,e^{-7x}dx$
7. $\int x^2\cos 5x\,dx$	8. $\int arctg\,2x\,dx$	9. $\int\cos(\ln x^2)dx$

Variant 9

1. $\int x\cos(2x-1)\,dx$	2. $\int\arccos 5x\,dx$	3. $\int\sqrt{x^2+4}dx$
4. $\int(x+1)e^{3x}\,dx$	5. $\int(1-2x)\ln(1-2x)\,dx$	6. $\int\sin 3x\,e^{-x}dx$
7. $\int(x^2+1)\sin 3x\,dx$	8. $\int xarcctgx\,dx$	9. $\int\ln\sqrt{x}\,dx$

Variant 10

1. $\int(1-2x)\cos 3x\,dx$	2. $\int\arcsin\sqrt{x}\,dx$	3. $\int\ln\sqrt{2x+1}dx$
4. $\int(7-x)e^{4x}\,dx$	5. $\int x^3\ln x\,dx$	6. $\int\cos 4x\,e^{2x}dx$
7. $\int(5-x^2)\sin x\,dx$	8. $\int arctg\,5x\,dx$	9. $\int\sin(\ln 5x)dx$

Variant 11

1. $\int (2+x)\cos x\,dx$	2. $\int x\arcsin x\,dx$	3. $\int \sqrt{x}\ln x\,dx$
4. $\int (x-5)e^{5x}\,dx$	5. $\int \frac{x}{\cos^2 x}dx$	6. $\int \cos x\,e^{5x}dx$
7. $\int (x-x^2)\cos 2x\,dx$	8. $\int arctgx\,dx$	9. $\int \cos(\ln x^2)dx$

Variant 12

1. $\int (3x+7)\sin 2x\,dx$	2. $\int \arccos 8x\,dx$	3. $\int \sin 6x\,e^{-x}dx$
4. $\int (5-x)e^{-x}\,dx$	5. $\int \frac{\ln x}{x^2}dx$	6. $\int \frac{dx}{\cos^3 x}$
7. $\int (1-3x^2)\sin 2x\,dx$	8. $\int x5^x\,dx$	9. $\int \sin(\ln x^2)dx$

Variant 13

1. $\int (3-2x)\sin x\,dx$	2. $\int x\arcsin 2x\,dx$	3. $\int \sqrt[3]{x}\ln x\,dx$
4. $\int (3x+1)e^{-4x}\,dx$	5. $\int x^4\ln x\,dx$	6. $\int \cos x\,e^{-7x}dx$
7. $\int (4-x^2)\cos x\,dx$	8. $\int arctg\,5x\,dx$	9. $\int \sqrt{x^2+1}dx$

Variant 14

1. $\int (x+4)\cos(3x-1)\,dx$	2. $\int x\arccos x\,dx$	3. $\int \sqrt{x^2+4}dx$
4. $\int (4x+1)e^{5x}\,dx$	5. $\int \sqrt{1-2x}\ln(1-2x)\,dx$	6. $\int \cos 3x\,e^{-x}dx$
7. $\int (x^2+2x)\sin 3x\,dx$	8. $\int arcctg\,4x\,dx$	9. $\int \ln\sqrt[3]{x}\,dx$

Variant 15

1. $\int (1-2x)\cos 7x\,dx$	2. $\int \arccos\sqrt{x}\,dx$	3. $\int \ln\sqrt{2x+1}dx$
4. $\int (8-3x)e^{-x}\,dx$	5. $\int x^5\ln x\,dx$	6. $\int \cos 3x\,e^{-2x}dx$
7. $\int (x-4x^2)\sin x\,dx$	8. $\int x\,arctgx\,dx$	9. $\int \sin(5\ln x)dx$

Variant 16

1. $\int (2-x)\sin x\,dx$	2. $\int x\arcsin x\,dx$	3. $\int \ln^2 x\,dx$
4. $\int (2x-4)e^{-5x3}\,dx$	5. $\int \frac{x}{\cos^2 x}dx$	6. $\int \sin 3x\,e^{2x}dx$
7. $\int x^2\cos 3x\,dx$	8. $\int arctg\,7x\,dx$	9. $\int \sin(\ln x^2)dx$

Variant 17

1. $\int (3x+7)\cos 2x\,dx$	2. $\int \arccos 7x\,dx$	3. $\int \sqrt[3]{x}\ln x\,dx$
4. $\int (5-x)5^{-x}\,dx$	5. $\int \frac{\ln x}{\sqrt{x}}dx$	6. $\int \sin x e^{-6x}dx$
7. $\int 3x^2\sin 3x\,dx$	8. $\int x^2e^x dx$	9. $\int \cos(\ln x^{2})dx$

Variant 18

1. $\int (3+5x)\sin 7x\,dx$	2. $\int x\arcsin 3x\,dx$	3. $\int arcctg\,5x\,dx$
4. $\int (3x+1)7^{-4x}\,dx$	5. $\int (2x+3)\ln x\,dx$	6. $\int \cos x e^{-7x}dx$
7. $\int (1+4x^2)\cos 2x\,dx$	8. $\int \frac{\ln x dx}{x^2}$	9. $\int \frac{x}{\cos^2 x}dx$

Variant 19

1. $\int (1-2x)\cos 2x\,dx$	2. $\int \arcsin\sqrt{x}\,dx$	3. $\int \ln^2 x\,dx$
4. $\int (4+3x)6^{-x}\,dx$	5. $\int x^5\ln x\,dx$	6. $\int \sin 3x 3^{2x}\,dx$
7. $\int (x+4x^2)\sin 3x\,dx$	8. $\int x arctgx\,dx$	9. $\int \sin(\ln x)dx$

Variant 20

1. $\int (3x+2)\sin x\,dx$	2. $\int \arcsin 5x\,dx$	3. $\int \frac{\ln x}{\sqrt{x}}dx$
4. $\int (2x+1)e^{2x}\,dx$	5. $\int \frac{1}{\cos^3 x}dx$	6. $\int \cos x 7^{2x}dx$
7. $\int (2x^2-1)\cos 3x\,dx$	8. $I=\int xarctgxdx$	9. $\int \ln^2 x\,dx$

§6. *Integration of fractional rational functions*

6.1. ***Factorization of Rational Function.*** Every polynomial of n degree, such as $Q_n(x)=a_0+a_1x+a_2x^2+...a_nx^n$ can be resolved into a product of factors, each of which is linear.

$$Q_n(x)=a_n(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n), \tag{18}$$

where α_i are the real or complex roots of the equation $Q_n(x) = 0$. Complex roots occur in pairs and are conjugated $\alpha = \lambda \pm \mu i$, where λ and μ are real numbers and i *is* imaginary unit ($i^2=-1,\ i=\sqrt{-1}$).

$$(x-\alpha)(x-\bar{\alpha})=\left[x-(\lambda+\mu i)\right]\left[x-(\lambda-\mu i)\right]=\left[(x-\lambda)-\mu i\right]\left[(x-\lambda)+\mu i\right]=$$
$$=(x-\lambda)^2+\mu^2=x^2-2\lambda x+\lambda^2+\mu^2=x^2+px+q, \quad \left(p=-2\lambda, \quad q=\lambda^2+\mu^2\right).$$

Thus, the products of complex conjugate pairs in (1) are the 2nd degree polynomials with real coefficients. If $Q_n(x)$ has m real roots and l pairs of complex roots the expansion (1) takes the form

$$Q_n(x)=a_n(x-\alpha_1)...(x-\alpha_m)(x^2+p_1x+q_1)...(x^2+p_lx+q_l). \quad (19)$$

If polynomial has some repeated roots it could be rewritten as

$$Q_n(x)=a_n(x-\alpha_1)^{r_1}...(x-\alpha_k)^{r_k}(x^2+p_1x+q_1)^{s_1}...(x^2+p_jx+q_j)^{s_j}. \quad (20)$$

6.2. Partial Fraction Decomposition. Let us consider functions where the numerator and denominator are polynomials. Such functions are called fractional rational functions and have the following form:

$$R(x)=\frac{P_m(x)}{Q_n(x)}=\frac{b_0+b_1x+b_2x^2+\cdots+b_mx^m}{a_0+a_1x+a_2x^2+\cdots+a_nx^n}. \quad (21)$$

A rational function is called ***proper*** if the degree of the numerator is less than the degree of the denominator, and ***improper*** otherwise. Thus, if $m < n$ then *R(x)* is a proper fractional rational function; if $n \leq m$ then *R(x)* is an improper fraction.

Any improper fraction can be expanded into a sum of a polynomial and a proper fraction by simple division.

$$R(x)=\frac{P_m(x)}{Q_n(x)}=H_{m-n}(x)+\frac{G_k(x)}{Q_n(x)},$$

where H_{m-n} is a polynomial of $n-m$ degree and $G_k(x)/Q_n(x)$ is a proper fraction.

The proper fraction $\dfrac{G_k(x)}{Q_n(x)}$ can be represented as a sum of partial proper fractions.

There are three cases to consider:

Case 1: if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real and unequal roots of the denominator $Q_n(x)$ then expansion into partial fractions takes on the following form:

$$\frac{A}{x-\alpha_1}+\frac{B}{x-\alpha_2}+\cdots+\frac{G}{x-\alpha_n}$$

Case 2: if α is repeated root of $Q_n(x)$ then r-fold linear factor $(x-\alpha)^r$ in the denominator corresponds r partial fractions of the form

$$\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+\cdots+\frac{A_r}{(x-\alpha)^r},$$

Case 3: if x^2+px+q has complex roots then factor $\left(x^2+px+q\right)^s$ corresponds:

$$\frac{M_1x+N_1}{x^2+px+q}+\frac{M_2x+N_2}{\left(x^2+px+q\right)^2}+\cdots+\frac{M_sx+N_s}{\left(x^2+px+q\right)^s},$$

where $A, B, G, A_1, A_2, \dots, M_1, N_1, \dots$ are constants to be determined.
Example,

$$\frac{x^3+2x^2-6}{x^3(x+1)\left(x^2+1\right)^2}=\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x^3}+\frac{B}{x+1}+\frac{M_1x+N_1}{x^2+1}+\frac{M_2x+N_2}{\left(x^2+1\right)^2}.$$

Now we need some ***method*** to determine constants $A_1, A_2, \dots, M_1, N_1$.

6.3. *How to Determine Coefficients?*

Suppose we have partial fraction decomposition:

$$\frac{P_m(x)}{Q_n(x)}=\frac{A_1}{(x-\alpha_1)^{r_1}}+\cdots+\frac{M_1x+N_1}{\left(x^2+px+q\right)^{s_1}}+\cdots \tag{22}$$

There are several methods for determining the coefficients for each term of (22) and we will go over each of those in the following examples. The most straightforward method is to multiply through by the common denominator $Q_n(x)$. We then obtain an equation of polynomials whose left-hand side (LHS) is simply $P_m(x)$ and whose right-hand side (RHS) has coefficients which are linear expressions of the constants $A_1, A_2, \dots, M_1, N_1, \dots$

Since two polynomials are equal if and only if their corresponding coefficients are equal, we can equate the coefficients of like terms. In this way, a system of linear equations is obtained which always has a unique solution. This solution can be found using any of the standard methods of linear algebra.

Example 1.

Decompose a rational fraction $\dfrac{5x^3-4x^2+12x-16}{x^4-16}$ in partial fraction.

Solution. This fraction is **proper**. So, the first step is to factor the denominator as much as possible and get the form of the partial fraction decomposition. Doing this gives:

$$\frac{5x^3-4x^2+12x-16}{(x-2)(x+2)\left(x^2+4\right)}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{Mx+N}{x^2+4}.$$

The next step is to add the right side back up and set numerators equal:

$$\frac{5x^3-4x^2+12x-16}{x^4-16}=\frac{A(x+2)\left(x^2+4\right)+B(x-2)\left(x^2+4\right)+(Mx+N)\left(x^2-4\right)}{x^4-16}. \tag{23}$$

Now open the brackets and collect all the like terms together.

$$5x^3-4x^2+12x-16=Ax^3+4Ax+2Ax^2+8A+Bx^3+4Bx-2Bx^2-8B+$$
$$+Mx^3-4Mx+Nx^2-4N$$

Then we will need to set the coefficients of like powers of x equal. This will give a system of equations that can be solved.

x^3	$A+B+M=5,$	From the 1st and 3rd equations: $A+B=4$,
x^2	$2A-2B+N=-4,$	from the 2nd and 4th: $A-B=-2$.
x^1	$4A+4B-4M=12,$	Solving the system:
x^0	$8A-8B-4N=-16.$	$\left.\begin{matrix} A+B=4 \\ A-B=-2 \end{matrix}\right\} \Rightarrow \quad A=1, B=3,$ so, $M=1, N=0.$

Therefore, $\dfrac{5x^3-4x^2+12x-16}{(x-2)(x+2)\left(x^2+4\right)}=\dfrac{1}{x-2}+\dfrac{3}{x+2}+\dfrac{x}{x^2+4}.$

Remark. It should be noticed that numerators of (23) must be equal for ***any*** x that we would choose to use. If the denominator of the LHS of (22) has real roots, it is convenient to substitute these values in the numerators. In such a way we determine some (or all) unknown coefficients and simplify system solution.

In Example 1, we can substitute $x=2$ and $x=-2$, in numerators of (23) and get $32A=32$, $A=1$and $-32B=-96$, $B=3$. Then we should use the standard method described above.

6.4. ***Integration of Partial Fractions.***

Decomposition of proper rational fraction could consist of four types of fractions:

$$\frac{A}{x-\alpha};\quad \frac{A}{(x-\alpha)^k}\ (k>1);\quad \frac{Ax+B}{x^2+px+q};\quad \frac{Ax+B}{\left(x^2+px+q\right)^k}\ (k>1).$$

The integral of the first two fractions are tabulated:

$$\int\frac{A}{x-\alpha}dx=A\ln|x-\alpha|+C;$$

$$\int\frac{A}{(x-\alpha)^k}dx=A\int(x-\alpha)^{-k}d(x-\alpha)=A\frac{(x-\alpha)^{1-k}}{1-k}+C.$$

The integration technique of function $\dfrac{Ax+B}{x^2+px+q}$ was described in Chapter 3.

Integral $I=\displaystyle\int\frac{Ax+B}{\left(x^2+px+q\right)^k}dx$, where $q>p^2/4$ could be integrated in the same way (create the differential of denominator in numerator and divide original fraction into two partial fractions), then for the second fraction we should use the recurrence formula:

$$I = \int \frac{dt}{\left(t^2 + a^2\right)^k} = \frac{1}{a^2(2k-2)}\left[\frac{t}{\left(t^2 + a^2\right)^{k-1}} + (2k-3)\int \frac{dt}{\left(t^2 + a^2\right)^{k-1}}\right]. \qquad (24)$$

According to formula (24) power of the denominator is reduced by 1. After *k-1* times of using (24) the integral is simplified to tabular form.

Example 1.

$$I = \int \frac{dx}{\left(x^2+1\right)^3} = \left\{\begin{matrix} \text{apply the formula } (24), \\ \text{where } k=3, \quad a^2=1 \end{matrix}\right\} =$$

$$= \frac{1}{4}\left[\frac{x}{\left(x^2+1\right)^2} + 3\int \frac{dx}{\left(x^2+1\right)^2}\right] = \left\{\begin{matrix} \text{apply the formula } (24), \\ \text{where } k=2, \quad a^2=1 \end{matrix}\right\} = \frac{x}{4\left(x^2+1\right)^2} +$$

$$+\frac{3}{4}\cdot\frac{1}{2}\left[\frac{x}{x^2+1} + \int \frac{dx}{x^2+1}\right] = \frac{\text{›}}{4\left(\text{›}^2+1\right)^2} + \frac{3}{8}\left(\frac{x}{x^2+1} + arctgx\right) + C.$$

Example 2.

$$I = \int \frac{2x^4 - 5x^3 - x^2 + 3x - 11}{\left(x^2-1\right)(x-3)}\,dx.$$

Solution. The integrand is an improper rational fraction (numerator is a polynomial of 4th power, denominator has the 3rd power).

The first step is to divide numerator by denominator. The result of such division is:

$$\frac{2x^4 - 5x^3 - x^2 + 3x - 11}{x^3 - 3x^2 - x + 3} = 2x + 1 + \frac{4x^2 - 2x - 14}{x^3 - 3x^2 - x + 3} = .$$

At the second step we need to factorize the denominator:

$$= 2x + 1 + \frac{4x^2 - 2x - 14}{(x-1)(x+1)(x-3)}.$$

At the next step we should decompose proper fraction into simple fractions:

$$\frac{4x^2 - 2x - 14}{(x-1)(x+1)(x-3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{D}{x-3}.$$

Then we need to find the coefficients A, B and D.

$$A(x+1)(x-3) + B(x-1)(x-3) + D(x-1)(x+1) = 4x^2 - 2x - 14.$$

The denominator of the original fraction has real roots $x_1 = 1$, $x_2 = -1$, $x_3 = 3$.

So,

$x = 1$ gives us the equality: $-4A = -12$, $A = 3$;

$x = -1$: $8B = -8$, $B = -1$;

$x = 3$: $8D = 16$, $D = 2$.

Therefore,

$$I = \int \left[2x + 1 + \frac{3}{x-1} - \frac{1}{x+1} + \frac{2}{x-3} \right] dx = x^2 + x + 3\,ln|x-1| - ln|x+1| + 2\,ln|x-3| + C =$$

$$= x^2 + x + \ln\left| \frac{(x-1)^3 (x-3)^2}{x+1} \right| + C.$$

Example 3.

Find $I = \int \frac{x\,dx}{x^3 - 8}$.

Solution. Fraction $x/(x^3 - 8)$ is proper. Factorization of the denominator and partial fraction decomposition give us:

$$\frac{x}{x^3 - 8} = \frac{x}{(x-2)(x^2 + 2x + 4)} = \frac{A}{x-2} + \frac{Bx + D}{x^2 + 2x + 4}.$$

This gives:

$$A(x^2 + 2x + 4) + (Bx + D)(x - 2) = x. \qquad (25)$$

If we substitute $x = 2$ we get $12A = 2$, $A = 1/6$.

To find B and D we need to compare coefficients of x^2 and x^0 in LHS and RHS of (25). This gives a set of linear equations:

$$\left. \begin{array}{l} A + B = 0 \\ 4A - 2D = 0 \end{array} \right\} \Rightarrow \quad B = -A = -1/6, \quad D = 2A = 1/3.$$

So, $I = \int \left[\frac{1/6}{x-2} + \frac{-1/6x + 1/3}{x^2 + 2x + 4} \right] dx = \frac{1}{6} \int \frac{dx}{x-2} - \frac{1}{6} \int \frac{x-2}{x^2 + 2x + 4} dx =$

$$= \frac{1}{6} ln|x-2| - \frac{1}{6} \int \frac{\frac{1}{2}(2x+2) - 3}{x^2 + 2x + 4} dx = \frac{1}{6} ln|x-2| - \frac{1}{12} \int \frac{2x+2}{x^2 + 2x + 4} dx + \frac{1}{2} \int \frac{dx}{(x+1)^2 + 3} =$$

$$= \frac{1}{6} ln|x-2| - \frac{1}{12} ln|x^2 + 2x + 4| + \frac{1}{2\sqrt{3}} arctg \frac{x+1}{\sqrt{3}} + C.$$

Individual Task 6

Variant 1

1. $\int \frac{2x^4+2x^3-41x^2+20\,dx}{x(x+5)(x-4)}$	2. $\int \frac{(2x^3+2x+1)\,dx}{(x^2-x+1)(x^{2+1)}}$	3. $\int \frac{x^3+4x^2+4x+2}{(x+1)^2(x^2+x+1)}dx$
4. $\int \frac{(x^3+6x^2+15x+2)dx}{(x-2)(x+2)^3}$	5. $\int \frac{(x^3+1)\,dx}{(x^2-x)}$	6. $\int \frac{x^3+6x^2+13x+9}{(x+1)(x+2)^3}dx$

Variant 2

1. $\int \frac{(3x^3+1)\,dx}{(x^2-1)}$	2. $\int \frac{(x^3+4x^2+2x+2)\,dx}{(x+1)^2(x^2+1)}$	3. $\int \frac{x^3-6x^2+14x-4}{(x+2)(x-2)^3}dx$
4. $\int \frac{(x^3+6x^2+13x+8)\,dx}{x(x+5)^3}$	5. $\int \frac{3x^4+3x^3-5x^2+2)\,dx}{x(x-1)(x+2)}$	6. $\int \frac{x^3+x+1}{(x^2-x+1)(x^2+1)}dx$

Variant 3

1. $\int \frac{(4x^4+2x^2-x-3)\,dx}{x(x-1)(x+1)}$	2. $\int \frac{(x^3+6x^2+10x+12)\,dx}{(x-2)(x+2)^3}$	3. $\int \frac{x^3-6x^2+11x-10}{(x-2)^3(x+2)}dx$
4. $\int \frac{(x^3+x^2+1)dx}{(x^2-x+1)(x^2+1)}$	5. $\int \frac{(x^3-17)dx}{x^2-4x+3}$	6. $\int \frac{2x^3+7x^2+7x-1}{(x+2)^2(x^2+x+1)}dx$

Variant 4

1. $\int \frac{(x^2+1)\,dx}{(x-1)^3(x+3)}$	2. $\int \frac{(x+4)\,dx}{(x^3+6x^2+11x+6)}$	3. $\int \frac{(2x+1)\,dx}{(x+1)(x-1)(x+2)}$
4. $\int \frac{(x+1)\,dx}{(x^2+1)(x^2+9)}$	5. $\int \frac{(2x^2-x+1)\,dx}{(x-2)^2(x^2+1)}$	6. $\int \frac{(x^3+x+1)\,dx}{(x^4-81)}$

Variant 5

1. $\int \frac{dx}{x^5-x^2}$	2. $\int \frac{x^2dx}{(x-1)^5}$	3. $\int \frac{(3x+1)\,dx}{(x+1)(x-3)(x+2)}$
4. $\int \frac{(x^3+3x^2+5x+7)\,dx}{(x^2+2)}$	5. $\int \frac{(3x^3+x^2+5x+1)\,dx}{(x^3+x)}$	6. $\int \frac{dx}{(x^2+2x+10)(x-2)}$

Variant 6

1. $\int \frac{(x^2+1)\,dx}{(x-1)^3(x+3)}$	2. $\int \frac{(x^5+1)\,dx}{(x^4-8x^2+16)}$	3. $\int \frac{(x+2)\,dx}{(x^4+16x^2)}$
4. $\int \frac{(x^3-2x)\,dx}{(x^2+1)^2}$	5. $\int \frac{dx}{(x^2-2x)^2}$	6. $\int \frac{x^4dx}{(x^4-16)}$

Variant 7

1. $\int \frac{(2x^3+5)\,dx}{(x^2-x-2)}$	2. $\int \frac{(2x^3+4x^2+2x-1)\,dx}{(x^2+2x+2)(x+1)^2}$	3. $\int \frac{(x^3+6x^2+18x-4)\,dx}{(x-2)(x+2)^3}$
4. $\int \frac{(x^3+6x^2+14x+10x)\,dx}{(x+1)(x+2)^3}$	5. $\int \frac{(2x^4-5x^2-8x-8)dx}{x(x^2-4)}$	6. $\int \frac{(2x^2-x+1)dx}{(x^2+x+1)(x^2+1)}$

Variant 8

1. $\int \frac{(2x^3-1)\,dx}{(x^2+x-6)}$	2. $\int \frac{(2x^3+6x^2+9x+6)\,dx}{(x^2+2x+2)(x+1)^2}$	3. $\int \frac{(x^3+6x^2+14x+4)\,dx}{(x-2)(x+2)^3}$
4. $\int \frac{(x^3-6x^2+11x+10x)\,dx}{(x+2)(x-2)^3}$	5. $\int \frac{(8-7x)dx}{(x-4)(x-2)(x+1)}$	6. $\int \frac{(3x^3+4x^2+6x)dx}{(x^2+2x+2)(x^2+2)}$

Variant 9

1. $\int \frac{(x^3-5x^2+5x+23)\,dx}{(x-1)(x+1)(x-5)}$	2. $\int \frac{(4x^2+3x+4)\,dx}{(x^2+1)(x^2+x+1)}$	3. $\int \frac{(x^3+6x^2+11x+7)\,dx}{(x+1)(x+2)^3}$
4. $\int \frac{(x^3+6x^2+4x+24)\,dx}{(x-2)(x+2)^3}$	5. $\int \frac{(3x^2+25)dx}{(x^2+3x+2)}$	6. $\int \frac{(2x^3+11x^2+16x+10)dx}{(x^2+2x+3)(x+2)^2}$

Variant 10

1. $\int \frac{(x^3+2x^2+3)\,dx}{(x-1)(x-2)(x-3)}$	2. $\int \frac{(3x^3+6x^2+5x-1)\,dx}{(x+1)^2(x^2+2)}$	3. $\int \frac{(2x^3+6x^2+5x+4)\,dx}{(x-2)(x+1)^3}$
4. $\int \frac{(2x^3+6x^2+7x+1)\,dx}{(x-1)(x+1)^3}$	5. $\int \frac{(-x^5+25x^3+1)dx}{(x^2+5x)}$	6. $\int \frac{(2x^3+7x^2+7x+9)dx}{(x^2+x+1)(x^2+x+2)}$

Variant 11

1. $\int \frac{(3x^3+2x^2+1)\,dx}{(x+2)(x-2)(x-1)}$	2. $\int \frac{(x^3+9x^2+21x)\,dx}{(x+3)^2(x^2+3)}$	3. $\int \frac{(2x^3+6x^2+7x)\,dx}{(x-2)(x+1)^3}$
4. $\int \frac{(x^3+6x^2+10x+10)\,dx}{(x-1)(x+2)^3}$	5. $\int \frac{(-x^5+9x^3+4)dx}{(x^2+3x)}$	6. $\int \frac{(2x^3+4x^2+2x+2)dx}{(x^2+x+1)(x^2+x+2)}$

Variant 12

1. $\int \frac{x^3\,dx}{(x+1)(x+2)(x-1)}$	2. $\int \frac{(x^3+6x^2+8x+8)\,dx}{(x+2)^2(x^2+4)}$	3. $\int \frac{(2x^3+6x^2+5x)\,dx}{(x+2)(x+1)^3}$
4. $\int \frac{(2x^3+6x^2+7x+2)\,dx}{x(x+1)^3}$	5. $\int \frac{(3x^5-12x^3-7)dx}{(x^2+2x)}$	6. $\int \frac{(x^2+x+3)dx}{(x^2+x+1)(x^2+1)}$

Variant 13

1. $\int \frac{(x^3-3x^2-12)\,dx}{(x-4)(x-3)(x-2)}$	2. $\int \frac{(x^3+5x^2+12x+4)\,dx}{(x+2)^2(x^2+4)}$	3. $\int \frac{(2x^3+6x^2+7x+4)\,dx}{(x+2)(x+1)^3}$
4. $\int \frac{(x^3-6x^2+13x-8)\,dx}{x(x-2)^3}$	5. $\int \frac{(2x^5-8x^3+3)dx}{(x^2-2x)}$	6. $\int \frac{(x^3+x+1)dx}{(x^2+x+1)(x^2+1)}$

Variant 14

1. $\int \frac{(x^3-3x^2-12)\,dx}{x(x-4)(x-3)}$	2. $\int \frac{(2x^3-4x^2-16x-12)\,dx}{(x-1)^2(x^2+4x+5)}$	3. $\int \frac{(2x^3+x+1)\,dx}{(x+1)x^3}$
4. $\int \frac{(x^3-6x^2+13x-7)\,dx}{(x+1)(x-2)^3}$	5. $\int \frac{(x^5+3x^3-1)dx}{(x^2+x)}$	6. $\int \frac{(2x^3+3x^2+3x+2)dx}{(x^2+x+1)(x^2+1)}$

Variant 15

1. $\int \frac{(3x^3+9x^2+10x+2)\,dx}{(x-1)(x+1)^3}$	2. $\int \frac{(-3x^3+13x^2-13x+1)\,dx}{(x-2)^2(x^2-x+1)}$	3. $\int \frac{(4x^3+x^2+2)\,dx}{x(x-1)(x-2)}$
4. $\int \frac{(x^3-6x^2+14x+6)\,dx}{(x+1)(x-2)^3}$	5. $\int \frac{(4x^3+24x^2+20x-28)dx}{(x^2+2x+2)(x+3)^2}$	6. $\int \frac{(x^5-x^3+1)dx}{(x^2-x)}$

Variant 16

1. $\int \frac{(3x^3-2)\,dx}{(x^3-x)}$	2. $\int \frac{(x^3+2x^2+10x)\,dx}{(x+1)^2(x^2-x+1)}$	3. $\int \frac{(x^3+x+2)\,dx}{(x+2)x^3}$
4. $\int \frac{(x^3-6x^2+10x+10)\,dx}{(x+1)(x-2)^3}$	5. $\int \frac{(x^3-3x^2-12)dx}{x(x-4)(x-2)}$	6. $\int \frac{(3x^3+x+46)dx}{(x^2+9)(x-1)^2}$

Variant 17

1. $\int \frac{2x^4+2x^3-41x^2+20\,dx}{x(x+5)(x-4)}$	2. $\int \frac{(x^3+6x^2+10x+12)\,dx}{(x-2)(x+2)^3}$	3. $\int \frac{(3x+1)\,dx}{(x+1)(x-3)(x+2)}$
4. $\int \frac{(x^3+6x^2+13x+8)\,dx}{x(x+5)^3}$	5. $\int \frac{(2x^2-x+1)\,dx}{(x-2)^2(x^2+1)}$	6. $\int \frac{x^4dx}{(x^4-16)}$

Variant 18

1. $\int \frac{(2x^3+5)\,dx}{(x^2-x-2)}$	2. $\int \frac{(4x^2+3x+4)\,dx}{(x^2+1)(x^2+x+1)}$	3. $\int \frac{(2x^3+6x^2+7x)\,dx}{(x-2)(x+1)^3}$
4. $\int \frac{(x^3-6x^2+11x+10x)\,dx}{(x+2)(x-2)^3}$	5. $\int \frac{(-x^5+25x^3+1)dx}{(x^2+5x)}$	6. $\int \frac{(x^2+x+3)dx}{(x^2+x+1)(x^2+1)}$

Variant 19

1. $\int \frac{(x^3-3x^2-12)\,dx}{(x-4)(x-3)(x-2)}$	2. $\int \frac{(-3x^3+13x^2-13x+1)\,dx}{(x-2)^2(x^2-x+1)}$	3. $\int \frac{(3x+1)\,dx}{(x+1)(x-3)(x+2)}$
4. $\int \frac{(x^3-6x^2+13x-7)\,dx}{(x+1)(x-2)^3}$	5. $\int \frac{(x^3-3x^2-12)dx}{x(x-4)(x-2)}$	6. $\int \frac{(2x^2-x+1)dx}{(x^2+x+1)(x^2+1)}$

Variant 20

1. $\int \frac{x^3+4x^2+4x+2}{(x+1)^2(x^2+x+1)}dx$	2. $\int \frac{(2x^2-x+1)dx}{(x^2+x+1)(x^2+1)}$	3. $\int \frac{(x^5+1)\,dx}{(x^4-8x^2+16)}$
4. $\int \frac{(x^3+x^2+1)dx}{(x^2-x+1)(x^2+1)}$	5. $\int \frac{(3x^3+x^2+5x+1)\,dx}{(x^3+x)}$	6. $\int \frac{(3x^3+4x^2+6x)dx}{(x^2+2x+2)(x^2+2)}$

§ 7. Integrating Some Irrational and Transcendental Functions

In this section the designation $R(u,v,...,w)$ indicates that only rational algebraic operations, actions of addition, subtraction, multiplication, division, raising to the integer power, are performed over the values u, v, ..., w. For example, a function $f(x)=(x+1)\Big/\left(1+\sqrt{(2x+1)^3}\right)$ should be classified as type $R\left(x,\sqrt{2x+1}\right)$, and function $f(x)=\left(e^x+1\right)\Big/\left(e^{2x}+4\right)$ – as type $R\left(e^x\right)$.

When integrating irrationality, if the integral is not tabular, the problem as a rule is to rationalize the integrand using suitable substitution. In some cases, it succeeds.

7.1. Integrals of the Type $I=\int R\left(x,\ \sqrt[n]{\frac{ax+b}{cx+d}}\right)dx$. (26)

Here n is a natural number, a, b, c, d - real constants, such as $ad \neq bc$.

To rationalize the integrand, we should use the substitution:

$$\frac{ax+b}{cx+d}=t^n. \qquad (27)$$

We solve for dx to give

$$ax+b=cxt^n+t^nd, \quad x=\frac{b-t^nd}{ct^n-a}, \quad dx=\frac{nt^{n-1}(ad-bc)}{\left(ct^n-a\right)^2}dt.$$

The integral becomes

$$I=\int R\left(\frac{b-t^nd}{ct^n-a},t\right)\frac{nt^{n-1}(ad-bc)}{\left(ct^n-a\right)^2}dt\cdot$$

The integrand becomes a rational function of argument *t*. Such integral in general case is an integral of rational fraction and could be resolved using techniques described in Chapter 6.

Example 1.

$$\int \frac{x\,dx}{1+\sqrt{x-1}} = \left\{ \begin{matrix} x-1=t^2, \;\; x=t^2+1 \\ dx=2tdt, \;\; t=\sqrt{x-1} \end{matrix} \right\} = \int \frac{\left(t^2+1\right)2t\,dt}{1+t} =$$

$$= 2\int \frac{t^3+t}{t+1}dt = 2\int\left(t^2-t+2-\frac{2}{t+1}\right)dt = 2\left(\frac{t^3}{3}-\frac{t^2}{2}+2t-2\ln|t+1|\right)+C =$$

$$= \frac{2}{3}\sqrt{(x-1)^3}-(x-1)+4\sqrt{x-1}-4\ln\left(1+\sqrt{x-1}\right)+C.$$

Example 2.

$$I = \int \frac{1}{x^2}\sqrt{\frac{x+1}{x}}\,dx = \left\{ \begin{matrix} \frac{x+1}{x}=t^2, \;\; x=\frac{1}{t^2-1} \\ dx=-\frac{2tdt}{\left(t^2-1\right)^2}, \;\; t=\sqrt{\frac{x+1}{x}} \end{matrix} \right\} =$$

$$= -\int (t^2-1)^2 t\frac{2tdt}{(t^2-1)^2} = -2\int t^2dt = -\frac{2}{3}t^3+C = -\frac{2}{3}\sqrt{\left(\frac{1+x}{x}\right)^3}+C.$$

Remark. In general case integral of the type:

$$\int R\left(x, \sqrt[m]{\frac{ax+b}{cx+d}}, \sqrt[n]{\frac{ax+b}{cx+d}}, \dots\right)dx$$

could be reduced to rational form using substitution $\frac{ax+b}{cx+d}=t^k$, where k – the least common multiple of $m, n, \ldots$

Example 3.

$$\int \frac{dx}{\sqrt{x+1}+\sqrt[3]{x+1}} = \left\{ \begin{matrix} x+1=t^6, \;\; x=t^6-1 \\ dx=6t^5dt, \;\; t=\sqrt[6]{x+1} \end{matrix} \right\} = \int \frac{6t^5dt}{t^3+t^2} =$$

$$= 6\int \frac{t^3\,dt}{t+1} = 6\int \frac{\left(t^3+1\right)-1}{t+1}dt = 6\int\left(t^2-t+1-\frac{1}{t+1}\right)dt = 6\left(\frac{t^3}{3}-\frac{t^2}{2}+t-ln|t+1|\right)+C =$$

$$= 2\sqrt{x+1}-3\sqrt[3]{x+1}+6\sqrt[6]{x+1}-6\ln\left(1+\sqrt[6]{x+1}\right)+C.$$

7.2. Integrals of the Type

$$I=\int R\left(x,\sqrt{bx^2+cx+g}\right)dx \quad (b\neq 0). \tag{28}$$

The integrand has a quadratic form under the square root sign. Completing the square in quadratic form reduces integral to one of the following types:

$$\begin{aligned} &а)\int R\left(u,\sqrt{a^2-u^2}\right)du;\\ &б)\int R\left(u,\sqrt{a^2+u^2}\right)du;\\ &в)\int R\left(u,\sqrt{u^2-a^2}\right)du. \end{aligned} \tag{29}$$

Rationalization of integrand is possible using the next trigonometric substitutions:

$$\begin{aligned} &а)\ u=a\sin t,\quad du=a\cos t\,dt;\\ &б)\ u=a\,tg\,t,\quad du=a\sec^2 t\,dt=\frac{adt}{\cos^2 t};\\ &в)\ u=a\sec t=\frac{a}{\cos t},\quad du=a\sec t\ tg\,t\,dt=\frac{atgtdt}{\cos t}. \end{aligned} \tag{30}$$

Example 4.

$$\int\sqrt{3+2x-x^2}\,dx=\int\sqrt{4-(x-1)^2}\,dx=\Big\{x-1=2\sin t,$$

$$dx=2\cos t\,dt,\ t=\arcsin\frac{x-1}{2}\Big\}=\int\sqrt{4-4\sin^2 t}\,\cdot 2\cos t\,dt=4\int\cos^2 t\,dt=$$

$$=2\int(1+\cos 2t)dt=2\left(t+\frac{1}{2}\sin 2t\right)+C=2t+2\sin t\cdot\sqrt{1-\sin^2 t}+C=$$

$$=2\,arcsin\frac{x-1}{2}+(x-1)\sqrt{1-\left(\frac{x-1}{2}\right)^2}+C.$$

Example 5.

$$\int\frac{dx}{\sqrt{\left(2+x^2\right)^3}}=\left\{\begin{aligned}&x=\sqrt{2}\,tgt,\quad t=arctg\left(\frac{x}{\sqrt{2}}\right)\\ &dx=\sqrt{2}\sec^2 t\,dt\end{aligned}\right\}=\int\frac{\sqrt{2}\sec^2 t\,dt}{\sqrt{\left(2+2tg^2 t\right)^3}}=$$

$$=\left\{1+tg^2 t=sec^2\,t,\ sec\,t=\frac{1}{cos\,t}\right\}=\frac{1}{2}\int\frac{sec^2\,t}{sec^3\,t}dt=\frac{1}{2}\int cos\,t\,dt=\frac{1}{2}sin\,t+C=$$

$$=\frac{1}{2}\sin\left(arctg\frac{x}{\sqrt{2}}\right)+C.$$

Remark. The next integral also belongs to the type described above.

$$I=\int\frac{dx}{(x-\alpha)\sqrt{bx^2+cx+g}} \qquad (31)$$

But in this case, substitution by inversion is more effective.

$$x-\alpha=\frac{1}{t}. \qquad (32)$$

Example 6.

$$\int\frac{dx}{(x-1)\sqrt{x^2-2x+2}}=\begin{Bmatrix} x-1=1/t,\ dx=-dt/t^2 \\ t=1/(x-1) \end{Bmatrix}=$$

$$=\int\frac{-1/t^2\,dt}{1/t\sqrt{1/t^2+1}}=-\int\frac{dt}{\sqrt{t^2+1}}=-ln\left|t+\sqrt{t^2+1}\right|+C=-ln\left|\frac{1}{x-1}+\sqrt{\frac{1}{(x-1)^2}+1}\right|+C.$$

7.3. Tangent Half-angle Substitution.

Another useful change of variables is the Weierstrass substitution, named after German mathematician Karl Weierstrass:

$$t=tg\frac{x}{2}, \qquad (33)$$

This substitution enables any rational function of the regular trigonometric functions to be integrated using the methods of partial fractions. The integral of the type $I=\int R(sin x,\ cos x)dx$ in the interval $(-\pi,\ \pi)$ could be converted to the integral of rational algebraic function of argument t. We will use the double-angle formula to replace *sinx*, *cosx*, and the differential *dx* with rational functions of a variable *t*.

$$\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}=2tg\frac{x}{2}\cos^2\frac{x}{2}=\frac{2tg(x/2)}{\sec^2(x/2)}=\frac{2tg(x/2)}{1+tg^2(x/2)}=\frac{2t}{1+t^2},$$

$$\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}=\cos^2\frac{x}{2}\left(1-tg^2\frac{x}{2}\right)=\frac{1-tg^2(x/2)}{1+tg^2(x/2)}=\frac{1-t^2}{1+t^2},$$

$$x=2arctgt,\ dx=\frac{2dt}{1+t^2}.$$

Therefore,

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}, \qquad dx = \frac{2dt}{1+t^2}. \tag{34}$$

This transforms a trigonometric integral into an algebraic integral, which may be easier to integrate: $I = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}$.

Example 7.

$$\int \frac{dx}{2\,sin\,x - cos\,x} = \left\{\begin{matrix} t = tg\frac{x}{2}, & sin\,x = \frac{2t}{1+t^2}, \\ cos\,x = \frac{1-t^2}{1+t^2}, & dx = \frac{2dt}{1+t^2} \end{matrix}\right\} =$$

$$= \int \frac{\frac{2dt}{1+t^2}}{\frac{4t}{1+t^2} - \frac{1-t^2}{1+t^2}} = 2\int \frac{dt}{t^2+4t-1} = 2\int \frac{d(t+2)}{(t+2)^2-5} = \frac{1}{\sqrt{5}}\,ln\left|\frac{t+2-\sqrt{5}}{t+2+\sqrt{5}}\right| + C =$$

$$= \frac{1}{\sqrt{5}}\,ln\left|\frac{tg\frac{x}{2}+2-\sqrt{5}}{tg\frac{x}{2}+2+\sqrt{5}}\right| + C.$$

The integral of type $I = \int R\left(sin^2 x, cos^2 x, tgx\right)dx$ could be transforms to rational algebraic form using the next substitution:

$$t = tgx. \tag{35}$$

In this case $sin^2 x = tg^2 x\, cos^2 x = \frac{tg^2 x}{sec^2 x} = \frac{tg^2 x}{1+tg^2 x}$, (36)

$$cos^2 x = \frac{1}{sec^2 x} = \frac{1}{1+tg^2 x}, \quad x = arctgt.$$

Therefore

$$tgx = t, \qquad sin^2 x = \frac{t^2}{1+t^2}, \qquad cos^2 x = \frac{1}{1+t^2}, \quad dx = \frac{dt}{1+t^2}. \tag{37}$$

The integral takes the following form:

$I = \int R\left(\frac{t^2}{1+t^2}, \frac{1}{1+t^2}, t\right)\frac{dt}{1+t^2} = \int R_1(t)dt$, where $R_1(t)$ – is rational function of argument t.

Example 8.

$$\int \frac{dx}{2-sin^2 x} = \left\{ t = tgx, \quad sin^2 x = \frac{t^2}{1+t^2}, \quad dx = \frac{dt}{1+t^2} \right\} =$$

$$= \int \frac{\frac{dt}{1+t^2}}{2-\frac{t^2}{1+t^2}} = \int \frac{dt}{t^2+2} = \frac{1}{\sqrt{2}} arctg \frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{2}} arctg\left(\frac{tgx}{\sqrt{2}}\right) + C.$$

7.4. Integrals of the type $I = \int R\left(e^x\right)dx$**.**

Substitution

$$t = e^x \tag{38}$$

leads to the integration of rational algebraic function $\frac{R(t)}{t}$.

Indeed, $t = e^x$, $x = lnt$, $dx = \frac{dt}{t}$ and $I = \int R(t)\frac{dt}{t}$.

Example 9.

$$\int \frac{e^{3x}dx}{e^{2x}+1} = \left\{ t = e^x, \quad dx = \frac{dt}{t} \right\} = \int \frac{t^3}{t^2+1}\cdot\frac{dt}{t} = \int \frac{t^2}{t^2+1}dt =$$

$$= \int \frac{\left(t^2+1\right)-1}{t^2+1}dt = \int\left(1-\frac{1}{t^2+1}\right)dt = t - arctgt + C = e^x - arctg\left(e^x\right) + C.$$

Individual Task 7

Variant 1

1. $\int \frac{(2x+3)dx}{\sqrt{2x-1}-\sqrt[4]{2x+3}}$	2. $\int \sqrt{4-x^2}dx$	3. $\int \frac{dx}{3\sin^2 x - 4\cos^2 x}$
4. $\int \frac{dx}{(x+2)\sqrt{x^2+2x+2}}$	5. $\int \frac{(\cos x - \sin x)dx}{(1+\sin x)^2}$	6. $\int \frac{2e^{2x}+7e^x}{e^{2x}+e^x-2}dx$

Variant 2

1. $\int \frac{(1-x)dx}{x\sqrt{3x+2}}$	2. $\int x^2\sqrt{9-x^2}dx$	3. $\int \frac{\sin^2 xdx}{1+\cos^2 x}$
4. $\int \frac{(x-1)dx}{(x+1)\sqrt{x^2+1}}$	5. $\int \frac{\cos xdx}{(1-\sin x)(1+\cos x)}$	6. $\int \frac{2e^{2x}-4e^x}{(e^x-2)(e^x-3)}dx$

Variant 3

1. $\int \frac{(\sqrt{1+x}-\sqrt[3]{1+x})dx}{\sqrt[3]{1+x}+4}$	2. $\int \frac{dx}{\sqrt{9+x^2}}$	3. $\int \frac{dx}{2\sin^2 x+9\cos^2 x}$
4. $\int \frac{(x-1)dx}{x\sqrt{2x^2-2x-1}}$	5. $\int \frac{\cos x dx}{(1-\sin x+\cos x)^2}$	6. $\int \frac{3e^{2x}+2e^x-3}{(e^{2x}-1)}dx$

Variant 4

1. $\int \frac{xdx}{1+\sqrt[3]{4x+1}}$	2. $\int \frac{dx}{\sqrt{4-x^2}}$	3. $\int \frac{dx}{1+4\cos^2 x}$
4. $\int \frac{(x-1)dx}{(x-1)\sqrt{3+2x-x^2}}$	5. $\int \frac{\sin x dx}{(1-\sin x+\cos x)^2}$	6. $\int \frac{e^x+2}{(e^{2x}-2e^x)}dx$

Variant 5

1. $\int x\sqrt{5x-1}dx$	2. $\int x^2\sqrt{16-x^2}dx$	3. $\int \frac{dx}{4-3\cos^2 x+5\sin^2 x}$
4. $\int \frac{dx}{x\sqrt{3x^2+x+1}}$	5. $\int \frac{\cos x dx}{(1+\sin x+\cos x)^2}$	6. $\int \frac{5e^{2x}-e^x}{(e^{3x}-3e^x-2)}dx$

Variant 6

1. $\int \frac{3x+4}{\sqrt[3]{x-1}+\sqrt{x-1}}dx$	2. $\int \frac{x^2}{\sqrt{2-x^2}}dx$	3. $\int \frac{dx}{1+\sin^2 x}$
4. $\int \frac{dx}{(x+1)\sqrt{x^2+2x+2}}$	5. $\int \frac{\sin x dx}{(1+\sin x)^2}$	6. $\int \frac{2e^{2x}-5e^x+1}{(e^{2x}-2e^x+1)}dx$

Variant 7

1. $\int \sqrt[3]{\frac{x-1}{x+1}}dx$	2. $\int \frac{\sqrt{x^2-1}}{x^4}dx$	3. $\int \frac{dx}{2+\cos^2 x}$
4. $\int \frac{dx}{x\sqrt{4x^2+6x-1}}$	5. $\int \frac{dx}{(5-3\cos x)}$	6. $\int \frac{5e^{2x}+2e^x}{(e^{2x}+2e^x+10)}dx$

Variant 8

1. $\int \frac{x+2}{x\sqrt{x-1}}dx$	2. $\int \frac{x^4}{\sqrt{(1-x^2)}}dx$	3. $\int \frac{dx}{1+2\cos^2 x+3\sin^2 x}$
4. $\int \frac{dx}{x\sqrt{3x^2-1}}$	5. $\int \frac{(1+\sin x)dx}{(1-\sin x)^2}$	6. $\int \frac{7e^x-15}{(e^{2x}-2e^x+5)}dx$

Variant 9

1. $\int \frac{x}{\sqrt{2x-1}+\sqrt[4]{2x-1}}dx$	2. $\int \frac{\sqrt{x^2-4}}{x^4}dx$	3. $\int \frac{tgxdx}{\sin^2 x+2\cos^2 x-3}$
4. $\int \frac{(2x+3)dx}{x\sqrt{6x-x^2}}$	5. $\int \frac{dx}{(5+4\sin x)}$	6. $\int \frac{2e^x}{(e^{3x}+8)}dx$

Variant 10

1. $\int \frac{\sqrt{3x+1}}{1+\sqrt[4]{3x+1}}dx$	2. $\int \frac{\sqrt{x^2-9}}{x^4}dx$	3. $\int \frac{dx}{3-\sin^2 x}$
4. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+5}}$	5. $\int \frac{dx}{(3+\cos x+2\sin x)}$	6. $\int \frac{e^x}{(e^x+1)(e^x-3)}dx$

Variant 11

1. $\int \frac{x+1}{\sqrt[3]{2x+1}}dx$	2. $\int x^2\sqrt{4-x^2}dx$	3. $\int \frac{tgxdx}{2\sin^2 x+3\cos^2 x-1}$
4. $\int \frac{(x+2)dx}{x\sqrt{3-x^2}}$	5. $\int \frac{dx}{(8+7\cos x-4\sin x)}$	6. $\int \frac{e^x}{(e^{4x}-1)}dx$

Variant 12

1. $\int \sqrt{\frac{x+1}{x}}dx$	2. $\int \frac{x^2}{\sqrt{1-x^2}}dx$	3. $\int \frac{(1+tgx)dx}{2\sin^2 x+3\cos^2 x}$
4. $\int \frac{(x-1)dx}{x\sqrt{2x^2-1}}$	5. $\int \frac{dx}{(3+\cos x-2\sin x)}$	6. $\int \frac{1}{(e^{3x}+3e^x)}dx$

Variant 13

1. $\int \frac{1}{(x+2)\sqrt{x+1}}dx$	2. $\int \frac{1}{\sqrt{(25+x^2)^3}}dx$	3. $\int \frac{(2-ctgx)dx}{\sin^2 x+4\cos^2 x}$
4. $\int \frac{(x-1)dx}{x\sqrt{4x^2+2x+1}}$	5. $\int \frac{dx}{(2+3\cos x)}$	6. $I=\int \frac{1}{(e^x+5)}dx$

Variant 14

1. $\int \frac{1}{(1+\sqrt[3]{x})\sqrt{x}}dx$	2. $\int \frac{x^2}{\sqrt{(4+x^2)^5}}dx$	3. $\int \frac{dx}{\sin^2 x+5\cos^2 x}$
4. $\int \frac{(x-1)dx}{(x+2)\sqrt{x^2+4x+8}}$	5. $\int \frac{\sin xdx}{(2+\sin x)}$	6. $\int \frac{e^x}{(e^{3x}-8)}dx$

Variant 15

1. $\int \frac{1}{\sqrt{2x+3}+\sqrt[3]{2x+3}}dx$	2. $\int \frac{1}{\sqrt{(2+x^2)^3}}dx$	3. $\int \frac{dx}{2\sin^2 x+\cos^2 x+3}$
4. $\int \frac{dx}{x\sqrt{4x^2-1}}$	5. $\int \frac{\sin xdx}{(5+3\sin x)}$	6. $\int \frac{e^x}{(e^{3x}+e^{2x}+2e^x+2)}dx$

Variant 16

1. $\int \frac{\sqrt[3]{x}}{\sqrt{x}+1}dx$	2. $\int x^2\sqrt{16-x^2}dx$	3. $\int \frac{(1+2tgx)dx}{\cos^2 x+1}$
4. $\int \frac{(2x+5)dx}{x\sqrt{4x-x^2-1}}$	5. $\int \frac{\cos xdx}{(1+\sin x-\cos x)}$	6. $\int \frac{e^x-5}{(e^{2x}+25)}dx$

Variant 17

1. $\int \frac{1}{\sqrt{x+2}+\sqrt[3]{x+2}}dx$	2. $\int \frac{dx}{\sqrt{(4+x^2)^3}}$	3. $\int \frac{(2tg^2x-11tgx-22)dx}{4-tgx}$
4. $\int \frac{(x-2)dx}{(x+1)\sqrt{x^2+2x}}$	5. $\int \frac{\cos xdx}{(5+4\cos x)}$	6. $\int \frac{1}{(e^{3x}+e^x)}dx$

Variant 18

1. $\int \frac{\sqrt[3]{x}}{\sqrt{x}-1}dx$	2. $\int \frac{x^4dx}{\sqrt{(8-x^2)^3}}$	3. $\int \frac{(4tgx-5)dx}{1-\sin^2 x+4\cos^2 x}$
4. $\int \frac{(2x+1)dx}{x\sqrt{9x-x^2}}$	5. $\int \frac{\cos xdx}{(2+\cos x)}$	6. $\int \frac{1}{(e^{3x}-e^x)}dx$

Variant 19

1. $\int \frac{1}{\sqrt{3x+1}+1}dx$	2. $\int \frac{x^2dx}{\sqrt{(1+x^2)^5}}$	3. $\int \frac{(6+tgx)dx}{9\sin^2 x+4\cos^2 x}$
4. $\int \frac{(4x+1)dx}{x\sqrt{x^2-6x+1}}$	5. $\int \frac{dx}{(3+3\cos x+5\sin x)}$	6. $\int \frac{e^x}{(e^{2x}-3)(e^{2x}+2)}dx$

Variant 20

$\int\sqrt{\frac{x}{4-x}}dx$	$\int\frac{\sqrt{x^2-1}dx}{x^4}$	$\int\frac{(\sin^2 x)dx}{3\cos^2 x-4}$
$\int\frac{(3x+1)dx}{x\sqrt{2x+x^2}}$	$\int\frac{\cos xdx}{(2+\cos x)}$	$\int\frac{1}{(e^x-2)}dx$

Questions for self-control

1. What change of variable should be made when integrating the expression $\int x\sqrt{3x^2+1}dx$: a) $t=x^2$; б) $t=\sqrt{3x^2+1}$; в) $t=3x^2$ г) $t=3x^2+1$? Take the reduced integral.
2. What change of variable should be made when integrating the expression $\int\frac{arctg2x}{1+4x^2}dx$: a) $t=arctg2x$; б) $t=x^2$; в) $t=4x^2$ г) $t=1/(1+4x^2)$? Take the reduced integral.
3. What change of variable should be made when integrating the expression $\int\frac{\sin 2x}{1-\cos 2x}dx$: a) $t=\sin 2x$; б) $t=\cos 2x$; в) $t=1-\cos 2x$ г) $t=1/(1-\cos 2x)$? Take the reduced integral.
4. What change of variable should be made when integrating the expression $\int\frac{1}{x\sqrt{1+\ln x}}dx$: a) $t=\ln x$; б) $t=1+\ln x$; в) $t=1/x$ г) $t=\sqrt{1+\ln x}$? Take the reduced integral.
5. What change of variable should be made when integrating the expression $\int\frac{e^{2x}}{1-2e^{2x}}dx$: a) $t=e^{2x}$; б) $t=-2e^{2x}$; в) $t=1-2e^{2x}$ г) $t=1/(1-e^{2x})$? Take the reduced integral.
6. What change of variable should be made when integrating the expression $\int\frac{\cos 3x}{\sqrt{2+\sin 3x}}dx$: a) $t=\sin 3x$; б) $t=\cos 3x$; в) $t=2+\sin 3x$ г) $t=\sqrt{2+\sin 3x}$? Take the reduced integral.
7. How should an integral of the form $\int(x-3)\sin 2xdx$ be integrated by parts? Which of the integrand functions should be choosen as u, and which one as dv in the formula $\int u\,dv=uv-\int v\,du$?
8. How should an integral of the form $\int\frac{x}{3}\ln(x-1)dx$ be integrated by parts? Which of the integrand functions should be choosen as u, and which one as dv in the formula $\int u\,dv=uv-\int v\,du$?
9. Integration by parts. Write down the basic formula. How should we integrate an integral of the form $\int(x+1)arctg2xdx$?

10. Integration by parts. Write down the basic formula. How should we integrate an integral of the form $\int e^x \sin 2x dx$?

11. Integration by parts. Write down the basic formula. How should we integrate an integral of the form $\int (2x+1)e^x dx$?

12. Integration by parts. Write down the basic formula. How should we integrate an integral of the form $\int \sin 2x e^{x/5} dx$?

13. How can we expand the fraction $\frac{x^3+2x}{(x^4-1)}$ by the sum of the elementary fractions? Choose the correct decomposition option:

а) $\frac{x^3+2x}{(x^4-1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x^2+1}$; б) $\frac{x^3+2x}{(x^4-1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$;

в) $\frac{x^3+2x}{(x^4-1)} = \frac{A}{x-1} + \frac{B}{x+1}$; г) $\frac{x^3+2x}{(x^4-1)} = \frac{A}{x-1} + \frac{Cx+D}{x^2+1}$?

14. How can we expand the fraction $\frac{x^2+2x}{(x-1)^3}$ by the sum of the elementary fractions? Choose the correct decomposition option: а) $\frac{x^2+2x}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x^2+1}$;

б) $\frac{x^2+2x}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{(x+1)^3}$; в) $\frac{x^2+2x}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)^3}$;

г) $\frac{x^2+2x}{(x-1)^3} = \frac{A}{x-1} + \frac{Bx+C}{(x-1)^2} + \frac{Dx+E}{(x+1)^3}$?

15. How can we expand the fraction $\frac{x^2+2x}{(x-1)^2(x^2+x+1)}$ by the sum of the elementary fractions? Choose the correct decomposition option:

а) $\frac{x^2+2x}{(x-1)^2(x^2+x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$; б) $\frac{x^2+2x}{(x-1)^2(x^2+x+1)} = \frac{A}{x-1} + \frac{C}{x^2+x+1}$;

в) $\frac{x^2+2x}{(x-1)^2(x^2+x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x^2+x+1}$;

г) $\frac{x^2+2x}{(x-1)^2(x^2+x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x^2+x+1}$?

16. How can we expand the fraction $\frac{x^2+2x}{(x-1)(x^2+x+1)^2}$ by the sum of the elementary fractions? Choose the correct decomposition option:

а) $\frac{x^2+2x}{(x-1)(x^2+x+1)^2}=\frac{A}{x-1}+\frac{Bx+C}{(x^2+x+1)^2}$; б) $\frac{x^2+2x}{(x-1)(x^2+x+1)^2}=\frac{A}{x-1}+\frac{Dx+E}{x^2+x+1}$;

в) $\frac{x^2+2x}{(x-1)(x^2+x+1)^2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}+\frac{Dx+E}{(x^2+x+1)^2}$; г) $\frac{x^2+2x}{(x-1)^2(x^2+x+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$?

17. How can we expand the fraction $\frac{x^2+2x}{(x+1)(2x+5)^2}$ by the sum of the elementary fractions? Choose the correct decomposition option:

а) $\frac{x^2+2x}{(x+1)(2x+5)^2}=\frac{A}{x+1}+\frac{B}{2x+5}+\frac{Cx+D}{(2x+5)^2}$; б) $\frac{x^2+2x}{(x+1)(2x+5)^2}=\frac{A}{x+1}+\frac{B}{2x+5}$;

в) $\frac{x^2+2x}{(x+1)(2x+5)^2}=\frac{A}{x+1}+\frac{B}{2x+5}+\frac{C}{(2x+5)^2}$; г) $\frac{x^2+2x}{(x+1)(2x+5)^2}=\frac{A}{x+1}+\frac{Cx}{(2x+5)^2}$?

18. How can we expand the fraction $\frac{x^2+2x}{(2x+1)^2(x-5)}$ by the sum of the elementary fractions? Choose the correct decomposition option:

а) $\frac{x^2+2x}{(2x+1)^2(2x-5)}=\frac{A}{2x+1}+\frac{B}{x-5}+\frac{Cx+D}{(2x+1)^2}$; б) $\frac{x^2+2x}{(2x+1)^2(2x-5)}=\frac{A}{2x+1}+\frac{B}{x-5}+\frac{D}{(2x+1)^2}$;

в) $\frac{x^2+2x}{(2x+1)^2(2x-5)}=\frac{A}{2x+1}+\frac{B}{x-5}+\frac{Cx}{(2x+1)^2}$; г) $\frac{x^2+2x}{(2x+1)^2(2x-5)}=\frac{A}{2x+1}+\frac{B}{x-5}$?

19. How can we expand the fraction $\frac{x^2+2x}{(x+7)(x-3)(x-2)}$ by the sum of the elementary fractions? Choose the correct decomposition option:

а) $\frac{x^2+2x}{(x+7)(x-3)(x-2)}= =\frac{A}{x+7}+\frac{B}{x-3}+\frac{C}{x-2}$; б) $\frac{x^2+2x}{(x+7)(x-3)(x-2)}=\frac{A}{x-3}+\frac{B}{x+7}+\frac{Cx+D}{x^2+1}$; в) $\frac{x^2+2x}{(x+7)(x-3)(x-2)}=\frac{A}{x-1}+\frac{B}{x+1}$;

г) $\frac{x^2+2x}{(x+7)(x-3)(x-2)}=\frac{A}{x-1}+\frac{Cx+D}{x^2+7}$?

20. How should the expressions of a kind $\int\cos^n x\sin^m x dx$ be integrated, if n is an even number, and m is an odd one?

21. How should the expressions of a kind $\int\cos^n x\sin^m x dx$, be integrated, if m is an even number, and n is an odd one?

22. How should the expressions of a kind $\int\cos^n x\sin^m x dx$, be integrated, if n and m is an odd one?

23. What substitution should be done while integrating an expression of the form $\int R(\sin x;\cos x;tgx)dx$?

24. What substitution should be done while integrating an expression of the form $\int R(\sin^2 x;\cos^2 x;tgx)dx$

25. What substitution is used when integrating the expressions of the form $\int R(\sin x;\cos x;tgx)dx$: а) $t=\sin x$; б) $t=tgx$; в) $t=\cos x$; г) $t=tg\frac{x}{2}$.

26. What substitution is used when integrating the expressions of the form $\int R(\sin^2 x;\cos^2 x;tgx)dx$: а) $t=\sin x$; б) $t=tgx$; в) $t=\cos x$; г) $t=tg\frac{x}{2}$.

27. What change of variable should be made when integrating the expression $\int R(x;\sqrt[n]{\frac{ax+b}{cx+d}})dx$: а) $t=\frac{ax+b}{cx+d}$; б) $t=\sqrt[n]{\frac{ax+b}{cx+d}}$; в) $t^n=\frac{ax+b}{cx+d}$?

28. What change of variable should be made when integrating the expression $\int R(x;\sqrt[n]{q^2-x^2})dx$: а) $x=q\cos t$; б) $x=q\sin t$; в) $x=qtgt$ г) $x=q\sec t$?

29. What change of variable should be made when integrating the expression $\int R(x;\sqrt[n]{q^2+x^2})dx$: а) $x=q\cos t$; б) $x=q\sin t$; в) $x=qtgt$ г) $x=q\sec t$?

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